BONN INTERNATIONAL GRADUATE SCHOOL OF MATHEMATICS

Clifford algebras, spin structures and twisted K-theory

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IMPRS Moduli Spaces

Introduction

Generalized cohomology theories can provide a wealth of invariants for topological spaces. Besides singular cohomology, one of the most well known cohomology theories is topological K-theory. The group $K^0(X)$ is defined as the group completion of the abelian monoid of vector bundles over X under direct sum $\oplus$. To define $K^n(X)$ for $n \neq 0$, one has to introduce more machinery.

Singular cohomology has the feature that any *oriented* manifold has a fundamental class that defines a pushforward or integral $\int_X : H^n(X) \to \mathbb{R}$. K-theory has a fundamental class only for *spin* manifolds, which is called the *spin orientation of K-theory.*

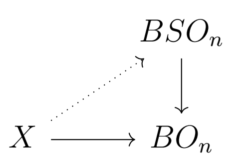
There is a third cohomology theory, topological modular forms TMF, which admits a fundamental class for all *string* manifolds. It is singled out among those three by the fact that there are no known cocycle models, even though there are various proposals, most notably the *Stolz-Teichner program*, which suggests two-dimensional supersymmetric quantum field theories as cocycles.

K-theory and spin geometry have been thoroughly investigated in the last 60 years. Both feature Clifford algebras prominently. On one side, Spin_n can be realized as a multiplicative subgroup of Cl_n. On the other side, some definitions of topological K-theory incorporate Clifford algebras explicitly to describe degree n-cocycles, whereas the original definition as the group completion of vector bundles can do without. The algebraic 8-periodicity of Clifford algebras matches the Bott periodicity of K-theory, but no direct implication is known.

We show below how Clifford algebras admit such a dual role in defining a) spin structures and b) twists of K-theory. Conjecturally, there is a similarly rich relationship between topological modular forms TMF and string geometry. As of yet, TMF and string structures only have a description in terms of homotopy theory. It is therefore important to recast some of the considerations in spin geometry and K-theory to be able to generalize them.

Orientations and the Whitehead tower

We will think of vector bundles over a manifold M in terms of their classifying map $\xi : M \to BO_n$ to the classifying space of the orthogonal group. A vector bundle is *orientable* if the classifying map lifts:



There is a short exact sequence of groups $SO_n \to O_n \xrightarrow{\det} \mathbb{Z}/2$ giving rise to a fiber sequence

$$BSO_n = \tau_{\geq 2} BO_n \to BO_n \xrightarrow{B\det} B\mathbb{Z}/2 = \tau_{\leq 1} BO_n \,.$$

More generally, BO_n sits in a fiber sequence that truncates the homotopy groups below and above a certain integer.

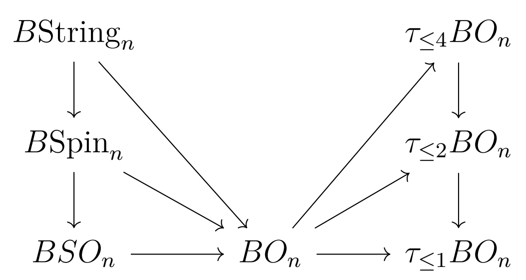


Figure 1: The Whitehead tower on the left and the Postnikov tower on the right are truncated versions of the space BO_n. The Postnikov tower is constructed by killing higher homotopy groups. The Whitehead tower is constructed as the homotopy fiber of the Postnikov tower, e.g. $B\mathrm{Spin}_n = \mathrm{fib}(BO_n \to \tau_{\leq 2} BO_n)$.

A vector bundle admits a *spin structure*, if its classifying map lifts to $B\mathrm{Spin}_n$. It admits a *string structure* if it lifts to $B\mathrm{String}_n$, etc. Since we are dealing with fiber sequences, we know the complete obstruction for such a lift: For example, a bundle $\xi : M \to BO_n$ is spin if and only if $M \to BO_n \to \tau_{\leq 2} BO_n$ is nullhomotopic. The cohomological obstruction one obtains here are the first and second Stiefel-Whitney classes.

Determinants and orientations

We can consider the topological category Vect of vector spaces. The space of objects of this category is topologized such that $\mathsf{Vect}^{\simeq} \cong \coprod_{n \in \mathbb{N}} BO_n$. We can define

$$\det : \mathsf{Vect} \longrightarrow \mathsf{Line}$$
$$V \longmapsto \det(V) := \bigwedge^{\text{top}} V \,.$$

- $\det(V)$ is $\otimes$-invertible
- Natural isomorphism $\det(V) \otimes \det(W) \xrightarrow{\cong} \det(V \oplus W)$
- The functor $\mathsf{Vect} \to \mathsf{Line}$ is monoidal, *but not symmetric!* Additional sign from determinant of switch map.
- The parity shifted determinant mapping to *super* lines sLine is symmetric monoidal. The category of super lines consists of one dimensional vector spaces with a parity. The necessary change is to add a *Koszul sign rule* for the symmetric braiding $\beta : V \otimes W \to W \otimes V$ mapping $v \otimes w \mapsto (-1)^{|v||w|} w \otimes v$.
- There are two isomorphism classes of super lines: the standard even line $\mathbb{R}$ and the parity shifted line $\Pi\mathbb{R}$. Moreover, we have:

$$\pi_0 \mathsf{sLine} = \{\mathbb{R}, \Pi\mathbb{R}\} \cong \mathbb{Z}/2$$
$$\pi_1 \mathsf{sLine} = \{\text{isometries } \mathbb{R} \to \mathbb{R}\} = \{\pm 1\} \cong \mathbb{Z}/2$$

The functor $\det : \mathsf{Vect} \to \mathsf{sLine}$ is the first stage of a categorical Postnikov tower.

Proposition 1. $\det : \mathsf{Vect} \to \mathsf{sLine}$ *is surjective on π_0 and an isomorphism on π_1. The higher homotopy groups of* sLine *vanish. On path components,* $\det : \mathsf{Vect}^{O_n} \to (\mathsf{sLine})_{\det(\mathbb{R}^n)}$ *is a $\pi_{\leq 1}$-isomorphism.*

We can take the categorical fiber of the functor det to *define* the category of oriented vector spaces:

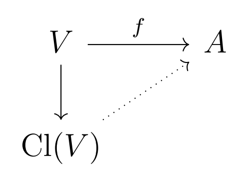
$$\mathsf{Vect}^{\mathrm{SO}_n} := \mathrm{fib}\left(\det : \mathsf{Vect}^{O_n} \to \mathsf{sLine}_{\det(\mathbb{R}^n)}\right) \,. \qquad (1)$$

- objects are pairs (V, or_V) where $V \in \mathsf{Vect}$ and $\mathrm{or}_V : \det(V) \xrightarrow{\cong} \det(\mathbb{R}^n)$
- morphisms are morphisms in Vect compatible with the orientations

Corollary 2. *The space of automorphisms of an object (V, or_V) in* $\mathsf{Vect}^{\mathrm{SO}_n}$ *is* SO_n.

Clifford algebras and the super Brauer group

Given a vector space V with inner product $\langle \cdot, \cdot \rangle$ we can generate the Clifford algebra $\mathrm{Cl}(V)$ with the following universal property: Every linear map $f : V \to A$ to an algebra A with $f(v)^2 = -\langle v, v \rangle \cdot 1$ extends uniquely to an algebra homomorphism $\mathrm{Cl}(V) \to A$:



Example 3. Let $\mathrm{Cl}_n = \mathrm{Cl}(\mathbb{R}^n)$ with the standard inner product.

Cl_0	Cl_1	Cl_2	Cl_3	Cl_4	Cl_5	Cl_6	Cl_7	Cl_8
$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$M_2(\mathbb{H})$	$M_4(\mathbb{C})$	$M_8(\mathbb{R})$	$M_8(\mathbb{R})^{\oplus 2}$	$M_{16}(\mathbb{R})$

The Clifford algebras carry a natural even/odd grading, i.e. they are *super algebras*. There is a symmetric monoidal bicategory sAlg_2 where the objects are super algebras, the 1-morphisms are bimodules and the 2-morphisms are bimodule homomorphisms. Composition is given by the relative tensor product of bimodules. The monoidal structure is the tensor product of algebras.

- $\mathrm{Cl}(V \oplus W) \cong \mathrm{Cl}(V) \otimes \mathrm{Cl}(W)$
- In sAlg_2, there is a (Morita) equivalence between Cl_n and Cl_{n+8}. This is the algebraic version of Bott periodicity.
- The functor $\mathrm{Cl} : (\mathsf{Vect}^O, \oplus) \to (\mathsf{sAlg}_2, \otimes)$ is symmetric monoidal and maps to $\otimes$-invertible algebras. This follows from $Cl_n \otimes Cl_{8-n} \simeq \mathbb{R}$.
- In fact, all $\otimes$-invertible super algebras are Morita equivalent to a Clifford algebra:

$$\pi_0 \mathsf{sAlg}_2^\times = \{\text{Morita eq. classes of inv. super algebras}\}$$
$$= \{\mathrm{Cl}_0, \ldots, \mathrm{Cl}_7\} \cong \mathbb{Z}/8 \text{ -the } \textit{super Brauer group}$$
$$\pi_1 \mathsf{sAlg}_2^\times = \{\text{invertible } \mathbb{R}\text{-}\mathbb{R}\text{-super bimodules}\} = \{\mathbb{R}, \Pi\mathbb{R}\} \cong \mathbb{Z}/2$$
$$\pi_2 \mathsf{sAlg}_2^\times = \{\text{isometries of bimodules } \mathbb{R} \to \mathbb{R}\} = \{\pm 1\} \cong \mathbb{Z}/2$$

Theorem 4 (DA)**.** $\mathrm{Cl} : \mathsf{Vect}^O \to \mathsf{sAlg}_2^\times$ *is surjective on π_0 and an isomorphism on π_1 and π_2. The higher homotopy groups of the right side vanish. On path components,* $\mathrm{Cl} : \mathsf{Vect}^{O_n} \to \left(\mathsf{sAlg}_2^\times\right)_{\mathrm{Cl}_n}$ *is a $\pi_{\leq 2}$-isomorphism.*

Geometric spin structures

So far, we defined spin structures in terms of a lifting. But we can repeat the trick we employed in definition 1 when defining the category of oriented vector spaces $\mathsf{Vect}^{\mathrm{SO}_n}$. By Theorem 4, we can similarly define $\mathsf{Vect}^{\mathrm{Spin}_n}$ as the categorical fiber of $\mathrm{Cl} : \mathsf{Vect}^{O_n} \to \mathsf{sAlg}_2$ over Cl_n:

$$\mathsf{Vect}^{\mathrm{Spin}_n} := \mathrm{fib}\left(\mathrm{Cl} : \mathsf{Vect}^{O_n} \to (\mathsf{sAlg}_2)_{\det(\mathbb{R}^n)}\right) \,.$$

An object of this category is a vector space V together with a Morita equivalence bimodule ${}_{\mathrm{Cl}(V)}\Sigma_{\mathrm{Cl}_n}$.

Corollary 5. *The space of automorphisms of an object (V, Σ) in* $\mathsf{Vect}^{\mathrm{Spin}_n}$ *is* Spin_n.

This explains the following definition:

Definition 6 ([ST04])**.** Let $V \to X$ be a vector bundle with inner product $\langle \cdot, \cdot \rangle$. A *geometric spin structure* on V is a Morita equivalence bimodule bundle ${}_{\mathrm{Cl}(V)}\Sigma_{\mathrm{Cl}(\underline{\mathbb{R}^n})}$ between the super algebra bundles $\mathrm{Cl}(V)$ and $\mathrm{Cl}(\underline{\mathbb{R}^n})$.

Remark 7. The Clifford linear Dirac operator of a spin manifold M is constructed directly from the bibundle ${}_{\mathrm{Cl}(TM)}\Sigma_{\mathrm{Cl}_n}$ and a compatible connection.

1 References

[ABS64] M. F. Atiyah, R. Bott, and A. Shapiro. Clifford modules. *Topology*, 3(suppl):3–38, 1964.

[ST04] Stephan Stolz and Peter Teichner. What is an elliptic object? In *Topology, geometry and quantum field theory*, volume 308 of *London Math. Soc. Lecture Note Ser.*, pages 247–343. Cambridge Univ. Press, Cambridge, 2004.

Application: twists and Thom classes

For a given vector bundle $\xi : M \to BO_n$ and for a given multiplicative cohomology theory E one can ask for a *Thom isomorphism* of $E^*(M)$-modules:

$$E^m(M) \overset{\cong}{\underset{?}{\to}} E_c^{m+n}(\xi)$$

If this exists, ξ is called *E-oriented.*

Example 8. The following table summarizes which bundles are oriented with respect to certain cohomology theories. HR denotes singular cohomology with R coefficients, KU and KO are complex and real topological K-theory and TMF is topological modular forms.

$H\mathbb{Z}/2$	$H\mathbb{Z}$	KU	KO	TMF
all	oriented	Spin^c	Spin	String

Homotopy theoretically, a bundle is E-oriented if and only if the following composite is nullhomotopic:

$$M \xrightarrow{\xi} BO_n \longrightarrow \mathbb{Z} \times BO \xrightarrow{J} \mathrm{Pic}(\mathbb{S}) \xrightarrow{-\otimes E} \mathrm{Pic}(E) \,.$$

Theorem 9 (DA)**.** *There is a homotopy equivalence* $\mathsf{sAlg}_2^\times \simeq \tau_{\leq 3}\mathrm{Pic}(KO)$.

- *We can identify the truncation of the twist*

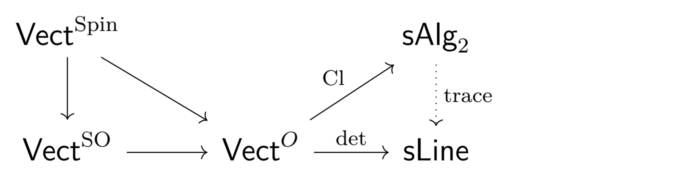
$$\mathbb{Z} \times BO \to \mathrm{Pic}(KO) \to \tau_{\leq 3}\mathrm{Pic}(KO)$$

with the group completion of the functor $\mathrm{Cl} : \mathsf{Vect}^O \to \mathsf{sAlg}_2$.

- *Hence, if a vector bundle is KO-oriented it must admit a spin structure.*

Remark 10. The second assertion is well-known and is usually obtained by quite different methods (c.f. [ABS64]). Also the converse holds.

To keep playing this game, it is important to see what the relation is between determinants and Clifford algebras. The answer is the following schematic:



(2)

Figure 2: The categorical version of the Whitehead tower.

Theorem 11 (DA)**.** *The functors in schematic 2 are well-defined and the diagram commutes.*

The functor trace is only defined on the core of the subcategory of dualizable objects. It is an instance of dimensional reduction or compactification of a TQFT. Concretely, it assigns to an algebra the cocenter $A/[A, A]$ and to a Morita equivalence ${}_A\Sigma_B$ its *Hattori-Stallings trace.*

There are a lot of open questions, most obviously: How does this categorical Whitehead tower continue to incorporate string structures? But also, on the level of K-theory there are open questions: Does topological Bott periodicity follow from the algebraic version? Is it possible to construct the E_∞ spin orientation $M\mathrm{Spin} \to ko$ by the above methods? And: How does this relate to twists of supersymmetric field theories in the Stolz-Teichner program?

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