Morse functions & h-principles

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1 Recollections

We will deal with Top-enriched[1] sheaves on the site of smooth n-manifolds and embeddings.

$$\Psi : \mathcal{M}\mathrm{fld}_n^{\mathrm{op}} \to \mathrm{Top}$$

Let $A \subset M$ be a closed subset. We can define the *space of germs around* A

$$\Psi(A \subset M) = \operatorname*{colim}_{A \subset U \text{ open}} \Psi(U)\,.$$

A boundary condition on A is an element $a \in \Psi(A \subset M)$. If $f \in \Psi(U)$, $U \subset M$ open, we let $\Psi(M \text{ rel } (U, f))$ be the pullback

$$\begin{tikzcd}
\Psi(M \text{ rel } (U,f)) \arrow[r] \arrow[d] & \Psi(M) \arrow[d] \\
* \arrow[r, "f"] & \Psi(U)
\end{tikzcd}$$

Let $A \subset M$ be closed. The *space of sections with boundary condition* $a \in \Psi(A \subset M)$ is the colimit

$$\Psi(M \text{ rel } a) = \operatorname*{colim}_{(U,f)} \Psi(M \text{ rel } (U,f))\,,$$

ranging over open neighborhoods $A \subset U$ and f whose germ at A is a.

Definition 1.1. Ψ is called *(micro)flexible* if for all $R \subset Q \subset M$ compact subsets, the restriction

$$\Psi(Q \subset M) \to \Psi(R \subset M)$$

is a Serre fibration (microfibration).

Example 1.2. Let Z be a topological space. The sheaf $M \mapsto C(M, Z)$ of continuous functions into Z is flexible. Similarly, sheaves of sections are flexible, e.g. Riemannian metrics.

Proof. We need to show that there is a lift

$$\begin{tikzcd}
\Delta^i \arrow[r, "f"] \arrow[d] & C(Q \subset M, Z) \arrow[d] \\
\Delta^i \times [0,1] \arrow[r, "F"] \arrow[ur, dotted] & C(R \subset N, Z)
\end{tikzcd}$$

[1]This is a lie, see Section 4.

We may assume that f is defined on an open U around Q and F is defined on V around R, see Section 4. We may further assume that $V \subset U$, by restricting the germ.

$$\begin{tikzcd}
\Delta^i \arrow[r, "f"] \arrow[d] & C(U,Z) \arrow[d] \\
\Delta^i \times [0,1] \arrow[r, "F"] \arrow[ur, dotted] & C(V,Z)
\end{tikzcd}$$

We are left to prove that restriction induces a Serre fibration on spaces of continuous functions. This diagram is adjoint to the diagram

$$\begin{tikzcd}
\Delta^i \times [0,1] \times V \sqcup_{\Delta^i \times \{0\} \times V} \Delta^i \times U \arrow[r] \arrow[d] & Z \\
\Delta^i \times [0,1] \times U \arrow[ur, dotted] &
\end{tikzcd}$$

This lift exists by the homotopy extension property for the pair $(U \times \Delta^i, V \times \Delta^i)$. $\square$

Theorem 1.3 (Gromov)**.** *Let Ψ be a microflexible sheaf. Let $L \subset K$ be compact, codimension zero submanifolds of M. Assume that K arises from L by attaching a k-handle, $k < n$. Then the restriction map*

$$\Psi(K \subset M) \to \Psi(L \subset M)$$

is a Serre fibration.

The main application of this is the following theorem which is proven by handle induction on handles *below the top dimension.*

Theorem 1.4. *Let Ψ be microflexible, Γ flexible and $j : \Psi \to \Gamma$ be a map that induces an equivalence on $\Psi(\mathbb{R}^n) \to \Gamma(\mathbb{R}^n)$. Then the map*

$$j : \Psi(M \ \mathit{rel}\ a) \to \Gamma(M \ \mathit{rel}\ j(a))$$

is a weak equivalence for all manifolds M, closed subsets $A \subset M$ such that $M \setminus A$ has ***no path component with compact closure in*** *M, and a a boundary condition on A.*

Furthermore, for every microflexible sheaf Ψ, one can construct a flexible sheaf Ψ^f with the above properties.

2 h-principles on closed manifolds

In [Kup19], Kupers works out conditions on Ψ that ensure that one can extend the h-principle to closed manifolds, which is the following theorem:

Theorem 2.1. *Suppose $j : \Psi \to \Gamma$ is a map of sheaves such that*

1. *Ψ is microflexible*
2. *Γ is flexible*

3. $j_{\mathbb{R}^n} : \Psi(\mathbb{R}^n) \to \Gamma(\mathbb{R}^n)$ *is a weak equivalence*

Let $A \subset M$ *be closed and* a *a boundary condition for* Ψ *on* A*. Consider*

$$j : \Psi(M \ \mathit{rel}\ a) \to \Gamma(M \ \mathit{rel}\ j(a)) \,. \tag{1}$$

1. If Ψ *satisfies (H), then* (1) *is a homology equivalence.*

2. If Ψ *satisfies (W), then* (1) *is a weak equivalence.*

We now define the conditions in terms of certain categories.

Definition 2.2. Let C^{Ψ} be the nonunital topological category with

- objects $|t, b|$, where $t > 0$ and b is a boundary condition on $S^{n-1} \times \{t\} \subset \mathbb{R}^n$.
- morphisms from $|t_0, b_0|$ to $|t_1, b_1|$ are given by $\Psi(S^{n-1} \times [t_0, t_1] \text{ rel } b_0 \sqcup b_1)$ for $t_0 < t_1$ and $\emptyset$ otherwise.

The nonunital category $\mathrm{Ho}(\mathsf{C}^{\Psi})$ has objects boundary conditions b on S^{n-1} and morphisms $\pi_0\Psi(S^{n-1}\times [1, 2] \text{ rel } b_1 \sqcup b_2)$.

The category $\mathrm{Ho}(\mathsf{C}^{\Psi})$ might have units. A boundary condition $b \in \Psi(S^{n-1})$ is *(left) fillable* if it extends to the disk.[2]

(H) Ψ satisfies (H) if $\mathrm{Ho}(\mathsf{C}^{\Psi})$ is a groupoid, i.e. has units and inverses.

(W) Ψ satisfies (W) if it satisfies (H) and if for every fillable class of boundary conditions, there exist representatives b_0, b_1, $t_0, t_1 > 0$ and an element $\iota \in \mathsf{C}^{\Psi}(|t_0, b_0|, |t_1, b_1|)$ such that all composition maps $\iota \circ -$, $- \circ \iota$ are weak equivalences.

Example 2.3. Flexible sheaves satisfy (H) and (W).

Proof. We prove the easier (H). To accomplish this we prove that the natural map $\mathsf{C}^{\Psi}(|t_0, b_0|, |t_1, b_1|) \to \mathrm{Paths}_{b_0,b_1}(\Psi(S^{n-1}))$ to the path space of the space of boundary conditions is an equivalence. This implies that the natural functor $\mathrm{Ho}(\mathsf{C}^{\Psi}) \to \Pi_1\Psi(S^{n-1})$ is an equivalence to a fundamental groupoid and Ψ hence satisfies (H).

Write $\mathcal{B} = \Psi(S^{n-1}) = \mathrm{colim}_n\, \Psi(S^{n-1} \times [-1/n, 1/n])$. (This is independent of the chosen $S^{n-1} \times \{t_0\}$ by using translations.) Note that restriction to t_0 defines a weak equivalence $\Psi(S^{n-1}\times[t_0, t_1]) \to \mathcal{B}$ since it is a filtered colimit of acyclic Serre fibrations.[3] Finally, the restriction to both endpoints $\Psi(S^{n-1} \times [t_0, t_1]) \to \mathcal{B} \times \mathcal{B}$ is a Serre fibration such that there is a commutative diagram

$$\begin{tikzcd}
\Psi(S^{n-1} \times [t_0, t_1]) \arrow[rr] \arrow[dr] & & \mathrm{Paths}(\mathcal{B}) \arrow[dl, "{(\mathrm{ev}_{t_0}, \mathrm{ev}_{t_1})}"] \\
& \mathcal{B} \times \mathcal{B} &
\end{tikzcd}$$

By 2-out-of-3 the horizontal map is a weak equivalence. On fibers, this induces the weak equivalence $\mathsf{C}^{\Psi}(|t_0, b_0|, |t_1, b_1|) \to \mathrm{Paths}_{b_0,b_1}(\mathcal{B})$. □

Let's sketch an idea of how to prove Theorem 2.1. Starting with a closed manifold, one can reduce to the case of manifolds with $S^{n-1} \subset \partial N \setminus A$ such that $N \setminus S^{n-1}$ is open:

[2] See the Erratum `https://www.utsc.utoronto.ca/people/kupers/wp-content/uploads/sites/50/2021/01/hprincipleserratum.pdf`

[3] Again, this uses that we are not in Top, but `qTop` or smooth sets: acyclicity is tested against the cofibrations $\partial P \to P$, where P is a compact manifold and we map this into a filtered colimit of acyclic Serre fibrations. We are done if this factors through a finite stage.

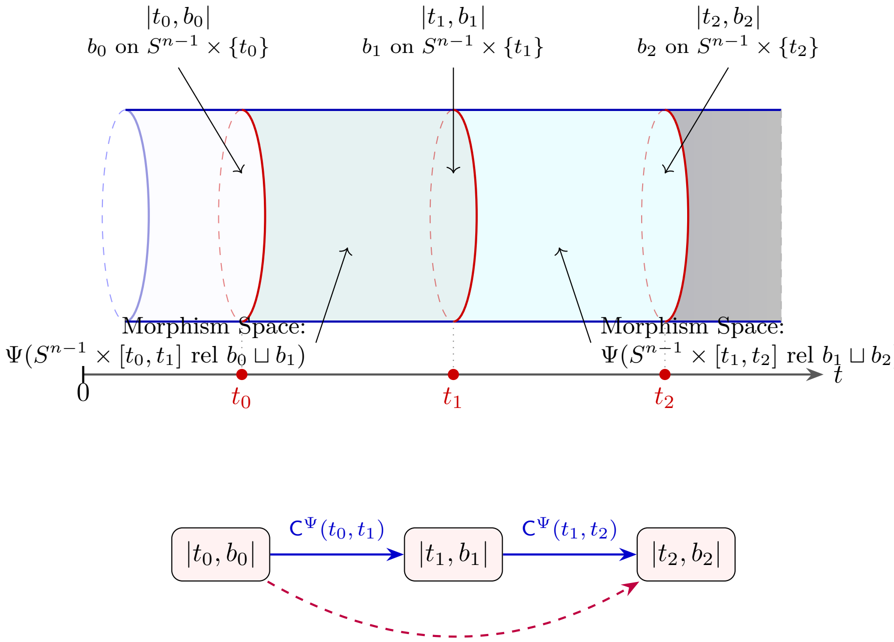

Figure 1: The category C^Ψ.

Resolving an infinite collar Let M be closed and choose $S^{n-1} \times \mathbb{R} \subset D^n \subset M$. There is a simplicial space $\Psi(M \text{ rel } a, D^n)_\bullet$ whose p-simplices are $t_0 \leq \cdots \leq t_p$, boundary conditions b_i on $S^{n-1} \times \{t_i\}$ and an element of

$$\Psi(M \setminus (S^{n-1} \times (t_0, t_p)) \text{ rel } b_0 \sqcup b_p \sqcup a) \times \prod_{i=0}^{p-1} \Psi(S^{n-1} \times [t_i, t_{i+1}] \text{ rel } b_i \sqcup b_{i+1})\,. \tag{2}$$

Kupers proves $\|\Psi(M \text{ rel } a, D^n)_\bullet\| \simeq \Psi(M \text{ rel } a)$ and the analogous claim for Γ. The levelwise (homology) equivalence of the simplicial spaces follows from the case with boundary S^{n-1} of Proposition 2.4, since all terms in (2) are of this form.

The following proposition is the only place where conditions (H) and (W) are crucially used.

Proposition 2.4. *$j : \Psi \to \Gamma$ as above. Let $A \subset M$ be closed and let $S^{n-1} \subset \partial M \setminus A$ such that $M \setminus S^{n-1}$ is open. Let b be a boundary condition on S^{n-1} and a a boundary condition on A. Consider*

$$j : \Psi(M \ \ rel\ a \sqcup b) \to \Gamma(M \ \ rel\ j(a) \sqcup j(b))\,. \tag{3}$$

1. *If Ψ satisfies (H) then* (3) *is a homology equivalence.*

2. *If Ψ satisfies (W) and b is fillable then* (3) *is a weak equivalence.*

Proof. We claim that under assumption (H) (or (W)) the homotopy fiber of

$$\Psi(M \cup_{S^{n-1}} S^{n-1} \times [0, \infty) \text{ rel } a) \to \Psi(S^{n-1} \times (0, \infty)) \tag{4}$$

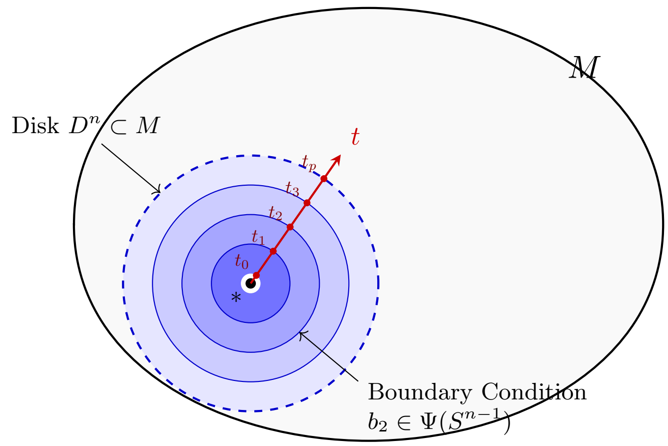

Figure 2: Resolving $\Psi(M)$ for a closed manifold M. The image depicts a p-simplex in $\Psi(M, D^n)_\bullet$.

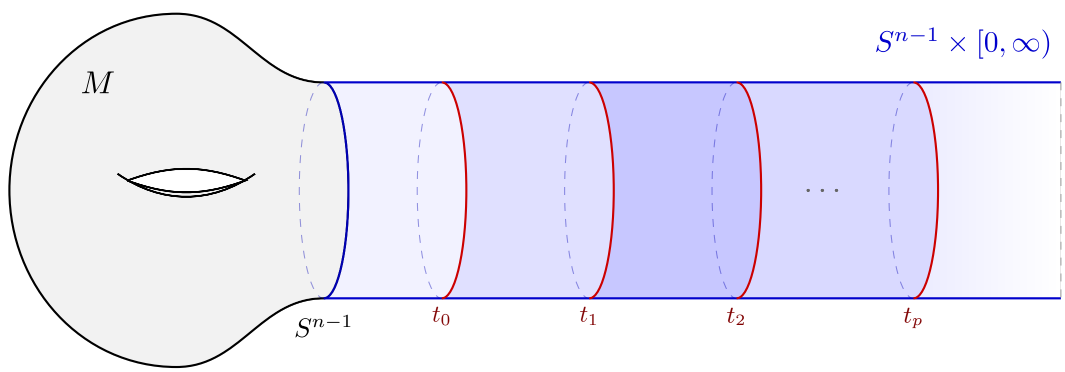

Figure 3: A manifold M with boundary S^{n-1} and a semi-infinite cylinder attached.

over b' is homology equivalent (weakly equivalent) to $\Psi(M \text{ rel } a \sqcup b)$ and the same for Γ. We can apply Gromov's h-principle to the open manifolds in (4) and deduce that (3) is a homology (weak) equivalence.

To prove the claim about fibers, one resolves both sides of (4) simplicially and notes that the simplicial space $\Psi(S^{n-1} \times (0, \infty))_\bullet$ is the geometric realization of C^Ψ. Similarly, $\Psi(M \cup S^{n-1} \times [0, \infty) \text{ rel } a)_\bullet$ is equivalent to a 1-sided bar construction, i.e. we can identify (4) with the homotopy fiber

$$\|B\mathsf{C}^\Psi\| \to \|B(F; \mathsf{C}^\Psi)\|, \tag{5}$$

where $F : \mathsf{C}^\Psi \to \text{Top}$ sends $|t, b|$ to $\Psi(M \cup_{S^{n-1}} S^{n-1} \times [0, t] \text{ rel } b)$. As a consequence of (H) ((W)), all the maps $g \circ - : \Psi(M \cup S^{n-1} \times [0, t]) \to \Psi(M \cup S^{n-1} \times [0, t + t'])$ are homology (weak) equivalences for $g \in \mathsf{C}^\Psi$. [Kup19, Lemma 43] implies that under this assumption the homotopy fiber of (5) over b is homology (weakly) equivalent to $F(|t, b|) \simeq_{H(W)} \Psi(M \text{ rel } a \sqcup b)$. $\square$

3 Morse and Cerf functions

Definition 3.1. Let $f : M \to \mathbb{R}$ be smooth. $p \in M$ is a critical point if $Df(p) = 0$. A critical point p of f is called nondegenerate if the Hessian $D^2 f(p)$ is an invertible matrix. f is a Morse function if all critical points of f are nondegenerate. The index of the critical point p is the dimension of the negative eigenspaces of $D^2 f(p)$.

Lemma 3.2 (Morse Lemma)**.** *Let f be Morse and let $p \in M$ be a critical point of index k. Then there are local coordinates (i.e. a local diffeomorphism to $\mathbb{R}^n$) such that f has the form*

$$f(x) = -\sum_{i=1}^{k} x_i^2 + \sum_{i=k+1}^{n} x_i^2 \,.$$

We want to consider the space of Morse functions. It is not connected.

Example 3.3. Examples of Morse functions on $\mathbb{R}$.

We have to incorporate singularities to make it connected.

Definition 3.4. A critical point p of f is called a *birth-death-singularity* of index $k + 1/2$ if f may locally be written as

$$f(x) = x_1^3 - \sum_{i=2}^{k} x_i^2 + \sum_{i=k+1}^{n} x_i^2 \,.$$

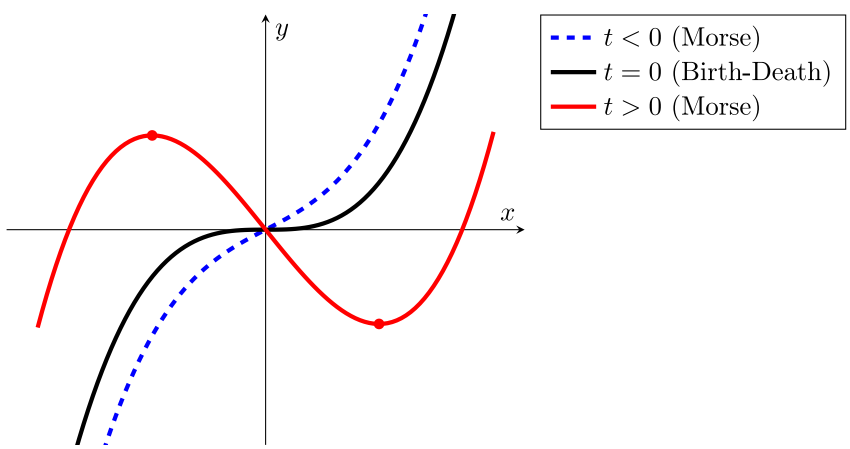

Figure 4: The family $\frac{1}{3}x^3 - tx$. Critical points collide at $t = 0$.

3.1 Jet Spaces

Definition 3.5. Let Z be a smooth manifold. We declare $f \sim_r g$ if $\partial^\alpha f(0) = \partial^\alpha g(0)$ for all multiindices α with $|\alpha| \leq r$. The rth jet space is $J^{(r)}(\mathbb{R}^n, Z) = C^\infty(\mathbb{R}^n, Z)/\sim_r$. The r-jet is the map $C^\infty(\mathbb{R}^n, Z) \to J^{(r)}(\mathbb{R}^n, Z)$.

For every fiber bundle $E \to M$ there is a jet bundle $J^r E \to M$ with fibers $J^{(r)}(\mathbb{R}^{\dim M}, E_x)$ and the r-jet map $j^r : \Gamma(E) \to \Gamma(J^r E)$.

Example 3.6. We can identify $J^{(r)}(\mathbb{R}^n,\mathbb{R})$ with the space $P^{(r)}$ of polynomials of degree $\le r$ via

$$f \mapsto \sum_{|\alpha|\le r} \frac{1}{\alpha!}\partial^\alpha f(0)x^{|\alpha|}\,.$$

Note that being Morse is a condition imposed on the second jet of the function.

Definition 3.7. A stratification of a subset $S \subset Y$ is a decomposition into a finite union of strata $\bigcup_{i=1}^n S_i$ which are locally closed submanifolds such that $\overline{S_k} = \bigcup_{i=k}^n S_i$.

There is a natural stratification of $J^{(r)}(\mathbb{R}^n,\mathbb{R})$, $r \ge 3$, which is invariant under the action of $\mathrm{Diff}(\mathbb{R}^n)$:

1. $df \neq 0$ The regular jets. This is an open dense submanifold of codimension 0.
2. The *Morse stratum* $\partial^1 f(0) = \cdots = \partial^n f(0) = 0$ but $\det \mathrm{Hess}_0(f) \neq 0$. These are n independent equations and an open condition, so that the codimension of this stratum is n.
3. The *Birth-Death stratum* $df(0) = 0$, $\det \mathrm{Hess}_0(f) = 0$, $\mathrm{rank}(\mathrm{Hess}_0(f)) = n-1$. This has codimension $n+1$.
4. Higher singularities: call this set $\mathcal{D}$. It is still stratified, but has codimension $\ge n+2$.

A $\mathrm{Diff}(\mathbb{R}^n)$-invariant stratification of $J^{(r)}(\mathbb{R}^n, Z)$ supplies stratifications of the bundle $J^r(M\times Z \to Z)$.

Theorem 3.8 (Thom's transversality theorem)**.** *Let $\mathcal{S}$ be a stratified subset of $J^{(r)}(\mathbb{R}^n,Z)$ that is* $\mathrm{Diff}(\mathbb{R}^n)$*-invariant. Then the set of smooth functions $g : M \to Z$ whose r-jet $j^r(g)$ is transverse to $\mathcal{S}$ is open and dense in $C^\infty(M,Z)$.*

Example 3.9.

- Morse functions are open and dense.

  Proof. Regard Morse functions as sections of trivial line bundle $M\times\mathbb{R}\to M$. Use theorem for $\mathcal{S}$ the closure of the birth and death stratum. This has codimension $n+1$. Differential of a section has rank n. Transversality of n-manifold and codimension $n+1$-manifold means that the intersection is empty, i.e. the section avoids the closure of Birth-Death stratum. □

- A generic path of Morse functions has birth-death singularities, but no higher ones.

  Proof. Same argument but over $M \times I$. □

Definition 3.10. Let $\mathcal{D} \subset J^{(r)}(\mathbb{R}^n,Z)$ be a $\mathrm{Diff}(\mathbb{R}^n)$-invariant subset. We let $\mathcal{F}(M,\mathcal{D})$ be the space of smooth functions $f : M \to Z$ whose r-jet avoids $\mathcal{D}$. Let $\mathcal{F}^f(M,\mathcal{D})$ be the subspace of sections of the bundle of r-jets of maps to Z which avoid $\mathcal{D}$.

Lemma 3.11. *If $\mathcal{D}$ is closed, then $\mathcal{F}(-,\mathcal{D})$ is microflexible.*

Proof. Let $R \subset Q$ be compact subsets of M. We want to construct a lift on a subinterval in the diagram

$$\begin{tikzcd} \Delta^i \arrow[r] \arrow[d] & \mathcal{F}(Q\subset M,\mathcal{D}) \arrow[d] \\ \Delta^i\times[0,1] \arrow[r] & \mathcal{F}(R\subset M,\mathcal{D}) \end{tikzcd}$$

By compactness, the maps to the germs will factor through a finite stage. We may also assume $V \subset U$.

$$\begin{array}{ccc} \Delta^i & \longrightarrow & \mathcal{F}(U,\mathcal{D}) \subset \mathcal{F}(U,\emptyset) \\ \big\downarrow & & \big\downarrow \\ \Delta^i \times [0,1] & \longrightarrow & \mathcal{F}(V,\mathcal{D}) \subset \mathcal{F}(V,\emptyset) \end{array}$$

There is a lift to $\mathcal{F}(U,\emptyset)$, which might hit $\mathcal{D}$. Since $\mathcal{D}$ is closed we will not hit it on a subinterval $[0,\epsilon]$. $\square$

Lemma 3.12. *Suppose $\mathcal{D}$ has strata of codimension $\geq n+2$. Then $\mathcal{F}(-,\mathcal{D})$ satisfies (H).*

Proof. We have already proved that the sheaf $C^\infty(-,Z)$ of smooth functions to Z is flexible and hence satisfies (H). By definition $\mathcal{F}(-,\mathcal{D}) \subset C^\infty(-,Z)$, and we will now prove that

$$\mathrm{Ho}(\mathsf{C}^{\mathcal{F}(-,D)}) \to \mathrm{Ho}(\mathsf{C}^{C^\infty(-,Z)})$$

is an inclusion of a full subcategory on the objects $|t,b|$, where b is a boundary condition of $\mathcal{F}$. A full subcategory of a groupoid is a groupoid and hence $\mathcal{F}(-,\mathcal{D})$ satisfies (H).

We will just show that

$$\pi_0\mathcal{F}(S^{n-1}\times[0,1] \text{ rel } b_0 \sqcup b_1, \mathcal{D}) \to \pi_0 C^\infty(S^{n-1}\times[0,1] \text{ rel } b_0 \sqcup b_1, Z)$$

is a bijection. It is surjective: any smooth function on $S^{n-1}\times[0,1]$ satisfying b_0 and b_1 near the boundary can be perturbed slightly relative boundary to be transverse to $\mathcal{D}$ by Thom's transversality. Because the codimension of $\mathcal{D}$ is too high ($n+2>n$) this means that the intersection is empty and it lies in $\mathcal{F}(S^{n-1},\mathcal{D})$. It is injective: Let H be a smooth homotopy between two functions lying in $\mathcal{F}(S^{n-1}\times[0,1],\mathcal{D})$ and regard it as a smooth function on $S^{n-1}\times[0,1]\times[0,1]$. Because the codimension is still $n+2>n+1$ a generic perturbation relative boundary will still yield a smooth homotopy that does not intersect $\mathcal{D}$. $\square$

The standard linear map $\Lambda : \mathbb{R}^n \to \mathbb{R}^z$ is the map $(x_1,\ldots,x_n) \mapsto (x_1,\ldots,x_z)$ if $z \leq n$ and $(x_1,\ldots,x_n,0,\ldots,0)$ for $z>n$.

Lemma 3.13. *Suppose $\mathcal{D}$ is invariant under* $\mathrm{Diff}(\mathbb{R}^z)$. [4] *Then the r-jet of Λ avoids $\mathcal{D}$, i.e. $\Lambda \in \mathcal{F}(\mathbb{R}^n,\mathcal{D})$ and any $b \in \mathcal{F}(S^{n-1})$ can be connected to $\Lambda|_{S^{n-1}}$.*

Proof. We claim that for all $a \in \mathbb{R}^z$ there is $f \in \mathcal{F}(\mathbb{R}^n,\mathcal{D})$ with $f(0)=a$ and $Df(0)$ of maximal rank. But r-jets of maximal rank have codimension 0 and we assumed translation invariance in $\mathbb{R}^z$, hence this exists.

Pick a and f as above. By the implicit function theorem there is a local diffeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ preserving the origin such that $f\circ\phi = \Lambda + a$. By $\mathrm{Diff}(\mathbb{R}^n)$-invariance of $\mathcal{D}$, we hence know that the r-jets of $\Lambda + a$ do not intersect $\mathcal{D}$. By $\mathrm{Diff}(Z)$-invariance, we also know this for Λ.

Note that $\pi_0\mathcal{F}(S^{n-1}\times[1,2] \text{ rel } b_0 \sqcup b_1) = *$ for all b_0, b_1. This follows from the proof of Lemma 3.12 since $Z=\mathbb{R}^z$ is contractible. $\square$

Lemma 3.14. *Let $\mathcal{D}$ be* $\mathrm{Diff}(\mathbb{R}^z)$*-invariant. $\mathcal{F}(-,\mathcal{D})$ satisfies (W).*

[4] weaker conditions also suffice

Proof. Again only $Z = \mathbb{R}^z$. Consider $\iota = \Lambda|_{S^{n-1}\times[1,2]}$. We prove that composition with $\iota \in \mathsf{C}^\Psi(|1, b_s|, |2, b_e|)$ induces a weak equivalence on mapping spaces in C^Ψ:

$$\iota_- = \iota \circ - : \mathcal{F}(S^{n-1} \times [t_0, 1], \mathcal{D} \text{ rel } b_0 \sqcup b_s) \to \mathcal{F}(S^{n-1} \times [t_0, 2], \mathcal{D} \text{ rel } b_0 \sqcup b_e)\,.$$

We can continuously retract the bigger cylinder onto the smaller. At time 1 this defines a map

$$r : \mathcal{F}(S^{n-1} \times [t_0, 2], \mathcal{D} \text{ rel } b_0 \sqcup b_e) \to \mathcal{F}(S^{n-1} \times [t_0, 1], \mathcal{D} \text{ rel } b_0 \sqcup b_s)\,.$$

See Figure 5 for a pictorial proof that this is a homotopy inverse. The last thing to check is that the reparametrizations avoid $\mathcal{D}$, but this is automatic by $\mathrm{Diff}(\mathbb{R}^n)$ and $\mathrm{Diff}(\mathbb{R}^z)$-invariance.

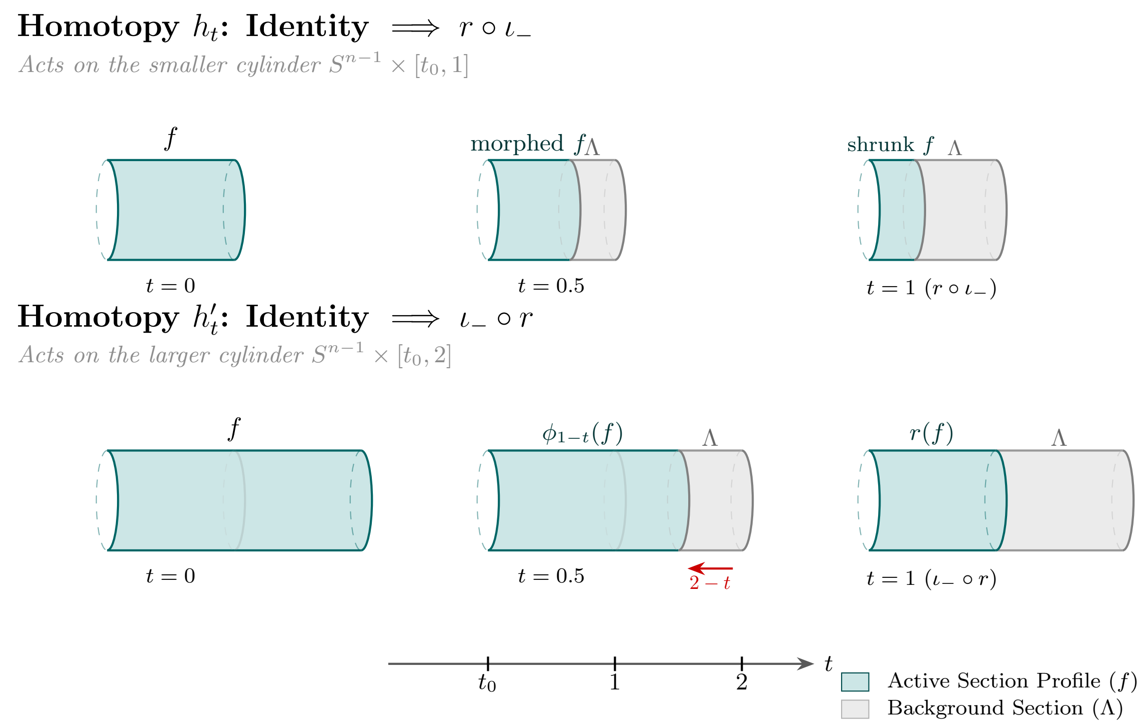

Figure 5: Depiction of the homotopies.

$\square$

Theorem 3.15. *Let Z be a smooth manifold and $\mathcal{D} \subset J^{(r)}(\mathbb{R}^n, Z)$ a $\mathrm{Diff}(\mathbb{R}^n)$-invariant closed stratified subset of codimension $\geq n+2$. Then $\mathcal{F}(-, \mathcal{D})$ satisfies a homological h-principle on closed manifolds, i.e.*

$$j^r : \mathcal{F}(M, \mathcal{D}) \to \mathcal{F}^f(M, \mathcal{D})$$

is a homology equivalence. If $\mathcal{D}$ is $\mathrm{Diff}(Z)$-invariant, then this is even a weak equivalence.

Proof. This is an application of the h-principle, Theorem 2.1. We have checked microflexibility in Lemma 3.11 and conditions (H) and (W) in Lemmas 3.12 and 3.14. We are left to show that $\mathcal{F}(\mathbb{R}^n, \mathcal{D}) \to \mathcal{F}^f(\mathbb{R}^n, \mathcal{D})$ is a weak equivalence. The right hand side is a space of sections of a trivial bundle over $\mathbb{R}^n$. This is weakly equivalent to its fiber $J^{(r)}(\mathbb{R}^n, Z) \setminus \mathcal{D}$ via evaluating at $0 \in \mathbb{R}^n$. We are left to prove that the r-jet at $0 \in \mathbb{R}^n$ induces a weak equivalence

$$j^r : \mathcal{F}(\mathbb{R}^n, \mathcal{D}) \to J^{(r)}(\mathbb{R}^n, Z) \setminus \mathcal{D}\,.$$

We will do the case $Z = \mathbb{R}^z$, where $J^{(r)}(\mathbb{R}^n, Z) \cong P^{(r)}(\mathbb{R}^n, \mathbb{R}^z)$ is the vector space of z-tuples of polynomials in n-variables of degree $\leq r$. We can reduce to showing the weak equivalence on the subspaces that take value $0 \in \mathbb{R}^z$ at the origin by restricting to the fibers of the Serre fibration $\mathrm{ev}_0 : \mathcal{F}(\mathbb{R}^n, \mathcal{D}) \to \mathbb{R}^z$. To prove weak equivalence we need to provide the dotted lift in the following diagram

$$\begin{tikzcd}
\partial\Delta^i \arrow[r, "f"] \arrow[d] & \mathcal{F}_0(\mathbb{R}^n, \mathcal{D}) \arrow[d, "j^r"] \\
\Delta^i \arrow[r, "F"] \arrow[ur, dotted] & P_0^{(r)}(\mathbb{R}^n, \mathbb{R}^z) \setminus \mathcal{D}
\end{tikzcd}$$

such that the upper triangle commutes and the lower triangle commutes up to specified homotopy relative $\partial\Delta^i$. Taylor approximation allows us to write any $g : \mathbb{R}^n \to \mathbb{R}^z$ as

$$g(x) = j^r(g)(x) + \sum_{|\alpha|=r+1} x^\alpha h_\alpha(x)\,,$$

where h_α is a smooth function on $\mathbb{R}^n$ with $\lim_{x\to 0} h_\alpha(x) = 0$. For $g \in \mathcal{F}_0(\mathbb{R}^n, \mathcal{D})$ we set

$$P_s(g) := j^r(g) + (1-s) \sum_{|\alpha|=r+1} x^\alpha h_\alpha\,.$$

This interpolates between g and the polynomial $j^r(g)$.

Note that for any $d \in \partial\Delta^i$, the r-jet of $P_s(f(d))$ at the origin is independent of s and coincides with the r-jet of $f(d)$ at the origin, which does not lie in $\mathcal{D}$. Therefore there is a small ball $B_{\epsilon(d)}(0)$, where the r-jet of $P_s(f(d))$ does not intersect $\mathcal{D}$ and we can choose $d \mapsto \epsilon(d)$ to be continuous.

Let $\eta(s, d)$ be a family of embeddings $\mathbb{R}^n \to \mathbb{R}^n$ parametrized by $s \in [0,1], d \in \partial\Delta^i$ such that

- $\eta(s, d)$ is the identity near the origin.
- $\eta(0, d)$ is the identity.
- $\eta(1, d)$ has image contained in $B_{\epsilon(d)}(0)$.

Under the identification $\Delta^i \cong \Delta^i \cup_{\partial\Delta^i \times \{2\}} \partial\Delta^i \times [0,2]$ define the lift L as

$$L(d') = \begin{cases} f(d) \circ \eta(d, s) & d' = (d, s) \in \partial\Delta^i \times [0,1] \\ P_{s-1}(f(d)) \circ \eta(d, 1) & d' = (d, s) \in \partial\Delta^i \times [1,2] \\ F(d') \circ \eta(d', 1) & \text{else} \end{cases}$$

The idea is: we can use F as a lift in a small ball and need to zoom in. Additionally we need to interpolate between f and its jet F at the boundary. The upper triangle commutes. The r-jets simplify to:

$$j^r L(d') = \begin{cases} j^r f(d) = F(d) & d' = (d, s) \in \partial\Delta^i \times [0,1] \\ j^r f(d) = F(d) & d' = (d, s) \in \partial\Delta^i \times [1,2] \\ j^r F(d') = F(d') & \text{else} \end{cases}$$

which is homotopic to $d' \mapsto F(d')$ by "pushing the boundary outward". Hence, the lower triangle commutes up to homotopy relative boundary. $\square$

We specialise to maps to the line, $Z = \mathbb{R}$, and the smallest set of singularities to which Theorem 3.15 applies.

Definition 3.16. The space of *generalized Morse functions* $\mathcal{G}(M)$ is the space of smooth functions $f : M \to \mathbb{R}$ all of whose critical points are Morse or birth-death singularities. Equivalently $\mathcal{G}(M) = \mathcal{F}(M, \mathcal{D})$, where $\mathcal{D} \subset J^{(3)}(\mathbb{R}^n, \mathbb{R})$ is the closed stratified subset of 3-jets of germs with a singularity worse than a birth-death singularity. We write $\mathcal{G}^f(M) = \mathcal{F}^f(M, \mathcal{D})$ for the associated space of *formal* generalized Morse functions, i.e. sections of the bundle $J^3(M \times \mathbb{R} \to M)$ avoiding $\mathcal{D}$.

The set $\mathcal{D}$ is invariant under $\mathrm{Diff}(\mathbb{R}^n)$ and $\mathrm{Diff}(\mathbb{R})$, and it has codimension $\geq n+2$: by the theorems of Morse and Cerf, generic functions have only Morse singularities and generic 1-parameter families have only Morse and birth-death singularities, so all worse singularities occur only in families of dimension $\geq n+2$. Theorem 3.15 therefore gives the following, originally due to Eliashberg–Mishachev (and proven in a range by Igusa, and on homology by Vassiliev).

Corollary 3.17 (Eliashberg–Mishachev)**.** *For every manifold M and boundary condition b for $\mathcal{G}$ near ∂M, the 3-jet map*

$$j^3 : \mathcal{G}(M \;\, rel\; b) \to \mathcal{G}^f(M \;\, rel\; j(b))$$

is a weak equivalence.

3.2 Computing the homotopy type

Since $\mathcal{G}^f$ is a space of sections, Corollary 3.17 reduces the homotopy type of $\mathcal{G}(M)$ to a bundle-theoretic computation, and the fibers of that bundle are governed by the local model $\mathcal{G}(\mathbb{R}^n)$. By the open h-principle (the case $M = \mathbb{R}^n$ in the proof of Theorem 3.15), evaluating the 3-jet at the origin gives a weak equivalence

$$\mathcal{G}(\mathbb{R}^n) \xrightarrow{\simeq} J_0^{(3)}(\mathbb{R}^n, \mathbb{R}) \setminus \mathcal{D},$$

the space of Morse-or-birth-death germs at 0, which by Example 3.6 is in turn the space of polynomials of degree ≤ 3 with such a singularity at the origin. This space was identified by Igusa.

Lemma 3.18 (Igusa)**.** *There is a weak equivalence $\mathcal{G}(\mathbb{R}^n) \simeq S^{n-1} * P$, where P is the homotopy pushout of the diagram*

$$\begin{tikzcd}[column sep=tiny]
& \frac{O(n)}{O(0)\times O(n-1)} \arrow[dl] \arrow[dr] & & \frac{O(n)}{O(1)\times O(n-2)} \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & \frac{O(n)}{O(n-1)\times O(0)} \arrow[dl] \arrow[dr] & \\
\frac{O(n)}{O(0)\times O(n)} & & \frac{O(n)}{O(1)\times O(n-1)} & & \frac{O(n)}{O(2)\times O(n-2)} & & \cdots & & \frac{O(n)}{O(n)\times O(0)}\,.
\end{tikzcd}$$

The bottom row records the Morse strata $\bar{A}_k$ of index k (sending f to the splitting $T_0\mathbb{R}^n = D_+ \oplus D_-$ into positive and negative eigenspaces of the Hessian), the top row records the birth-death strata $B_{k+1/2}$ of index $k + \frac{1}{2}$, and the maps are the standard inclusions of Grassmannians.

Sketch. Writing $E_3^{\mathcal{G}}(\mathbb{R}^n)$ for the polynomials whose value and first derivative vanish at the origin, one splits off the value and gradient to obtain $\mathcal{G}(\mathbb{R}^n) \simeq S^{n-1} * E_3^{\mathcal{G}}(\mathbb{R}^n)$. The Hessian decomposes $E_3^{\mathcal{G}}(\mathbb{R}^n)$ as the pushout of the strata $\bar{A}_k$ and $B_{k+1/2}$ above, and recording the eigenspace splitting identifies each stratum with the indicated homogeneous space via a fibration with convex (hence contractible) fibers. The inclusions $B_{k+1/2} \hookrightarrow \bar{A}_k$ and $B_{k+1/2} \hookrightarrow \bar{A}_{k+1}$ become the standard Grassmannian inclusions. $\square$

In low dimensions this can be evaluated by hand.

Example 3.19. For $n = 1$ all four corner terms $O(1)/(O(0) \times O(1))$ and $O(1)/(O(1) \times O(0))$ are a point, while $B_{1/2} = O(1)/(O(0) \times O(0)) = O(1) = S^0$. Hence P is the homotopy pushout of $* \leftarrow S^0 \rightarrow *$, i.e. $P \simeq S^1$, the two points of S^0 recording a local minimum and a local maximum. Therefore

$$\mathcal{G}(\mathbb{R}) \simeq S^0 * S^1 = S^2 .$$

A generator of $\pi_2\mathcal{G}(\mathbb{R})$ is the family $v(\epsilon, t, \theta) = tx + \sin(\theta)x^2 + \cos(\theta)x^3$, with $\epsilon \in S^0$, $t \in [0,1]$ and $\theta \in S^1$. Since TS^1 is trivial, the same computation gives $\mathcal{G}(S^1) \simeq S^2$.

4 Appendix

We sometimes needed the following fact: Let K be a compact manifold. Let $A \subset M$ be closed. Then any map $K \to \Psi(A \subset M)$ factors through a $\Psi(U)$ for some open neighborhood U of A in M.

While the diagram indexing the space of germs $\Psi(A \subset M)$ is filtered, it is not generally true that $\mathrm{Hom}_{\mathrm{Top}}(K, -)$ commutes with filtered colimits. There are some workarounds, all of which exchange Top for another category of sheaves on some site:

- Historically, Gromov worked with quasitopological spaces qTop. In modern parlance, qTop is the category of concrete sheaves on compact Hausdorff spaces.
- Diffeological spaces, which are concrete sheaves on the site of smooth manifolds (and all smooth maps).
- Smooth Sets, which are sheaves of sets on the site of smooth manifolds (and all maps). This is what Kupers uses in [Kup19].

See also https://mathoverflow.net/q/124616.

References

[Kup19] Alexander Kupers. Three applications of delooping to h-principles. *Geom. Dedicata*, 202:103–151, 2019.