Lecture notes Dan Berwick-Evans: Elliptic objects and supersymmetric field theories

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1 Lecture 1

Conjecture 1.1 (Segal, Stolz-Teichner)**.** *2-dimensional supersymmetric field theories determine geometric cocycles for (twisted, equivariant) elliptic cohomology.*

No QFT today, difficult to grasp in one shot. Prototype: supersymmetric quantum mechanics gives a geometric model for (twisted, equivariant) K-theory.

1.1 Quantum mechanics

Definition 1.2. The groupoid QM has objects: Hilbert spaces $\mathcal{H}$ equipped with an unbounded self-adjoint operator $H : \mathcal{H} \to \mathcal{H}$. Morphisms are $[U] \in PU(\mathcal{H}, \mathcal{H}')$ such that $H' = UHU^{-1}$.

Terminology

- H is the Hamiltonian generating time evolution e^{itH}; under Wick rotation this becomes e^{-tH}.
- Vectors in $\mathcal{H}$ are *states*.
- Eigenvalues of H are *energy levels*.
- The λ-eigenspace of H is the collection of states of energy λ.
- *Ground states* are the lowest-energy states.

Example 1.3. If M is an oriented Riemannian manifold, take $\mathcal{H} = L^2(M)$ and $H = \Delta$ the Laplacian. Ground states are harmonic functions. Then $(L^2(M), \Delta) \in \mathsf{QM}$. This theory is compact if and only if M is compact. Isometries $f : M \to M'$ induce morphisms $[f^*] : (L^2(M'), \Delta) \to (L^2(M), \Delta)$ in QM.

Definition 1.4. $(\mathcal{H}, H) \in \mathsf{QM}$ is *compact* if $\mathrm{Tr}(e^{-tH}) < \infty$ for all $t > 0$.

Compactness implies that the spectrum of H is discrete, bounded below and each eigenspace is finite dimensional.

From now on: assume compactness and $H \geq 0$ nonnegative.

Definition 1.5. A quantum mechanical system with G-symmetry is a functor $BG \to \mathsf{QM}$. Unpack this: $* \mapsto (\mathcal{H}, H)$ and $\rho : G \to PU(\mathcal{H})$ such that $H = \rho(g)H\rho(g)^{-1}$ for all $g \in G$. This lifts to a representation of a central extension $\hat{G}$ of G on $\mathcal{H}$.

If G is a topological group, this homomorphism should be continuous with respect to a topology Dan will specify later.

Definition 1.6. The *RG-flow* is the $\mathbb{R}_{>0}$-action on QM given by $(\mathcal{H}, H) \mapsto (\mathcal{H}, \lambda H)$ for $\lambda > 0$.

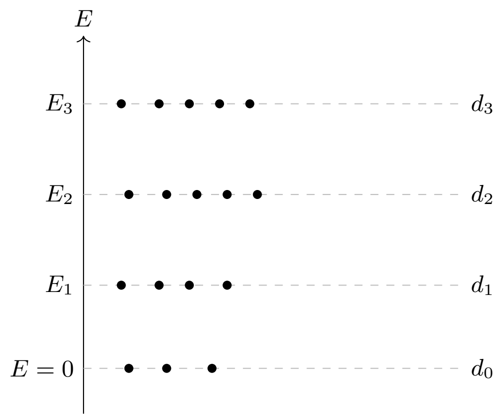
Definition 1.7. A quantum mechanical system is *topological* if $H = 0$.

$(\mathcal{H}, H)$ is topological if and only if it is fixed under the RG-flow. The limits of the RG-flow for $\lambda \to 0$ and $\lambda \to \infty$ are called UV-limit and IR-limit, respectively. We will have to define in what sense these are limits.

Example 1.8. For quantum mechanics on M, $(L^2(M), \Delta)$ is topological if and only if M is discrete. If $\dim(M) > 0$, then the RG-flow is nontrivial and the IR-limit is a topological theory determined by $\ker(\Delta) = \mathbb{C}^{\pi_0(M)}$. The UV-limit is $(L^2(M), 0)$ which is noncompact in positive dimensions.

1.2 QM as a topological category

Picture of a Hamiltonian:



The RG-flow is a rescaling of the vertical axis.

One can identify the space of Hamiltonians with the space of configurations of points on the positive real line, where each point is labeled by a finite-dimensional Hilbert space. Hohnhold, Stolz and Teichner [?] define a topology on Hamiltonians. We endow the set of configurations $\mathrm{Conf}(\mathcal{H})$ with a topology that allows eigenspaces to merge and split. embeddings of eigenspaces can change continuously and eigenspaces can move to infinity.

In $\mathrm{Conf}(\mathcal{H})$ topological theories are encoded by a finite dimensional $V \subset \mathcal{H}$ with the interpretation that $V^\perp$ is the eigenspace at infinity.

Theorem 1.9 (Hohnhold-Stolz-Teichner)**.** $\mathrm{Conf}(H)$ *is contractible. Also* $QM \simeq \mathrm{Conf}(\mathcal{H})/PU(\mathcal{H})$ *and hence* $B(\mathsf{QM}) \simeq K(\mathbb{Z}, 3)$.

Remark 1.10. There is a 2-category $2QM$ of quantum mechanical systems, where 2-morphisms are unitary intertwiners. ask DAN, related to $B^2U(1) = BPU(H)$

1.3 Supersymmetric quantum mechanics

> *Signs for super Hilbert spaces are confusing. You should ask Luuk instead of tying a knot in your brain.*

Definition 1.11. A $\mathbb{Z}/2$-graded Hilbert space is a Hilbert space $\mathcal{H}$ with an orthogonal decomposition $\mathcal{H} = \mathcal{H}^{even} \oplus \mathcal{H}^{odd}$. An operator $u : \mathcal{H} \to \mathcal{H}$ is homogeneous if it preserves the grading or swaps the grading. The graded projective unitary group $PU^{\pm}(\mathcal{H}) = \{u \in PU(\mathcal{H}) \text{ homogeneous}\}$ is the graded group of projective unitary operators.

Remark 1.12. The terminology is that *super* refers to a braided monoidal structure. To avoid technicalities Dan wants to not use this word. The problem is defining positivity of inner products on super vector spaces and self-adjointness of operators.

Definition 1.13. The groupoid SQM has objects $(\mathcal{H}, Q)$ where Q is an odd unbounded self-adjoint operator Q. The morphisms are $u \in PU^{\pm}(\mathcal{H}, \mathcal{H}')$ such that $Q' = uQu^{-1}$. The compactness conditon is that $Tr(e^{-tQ^2}) < \infty$ for all $t > 0$.

There is a functor $\mathsf{SQM} \to \mathsf{QM}$ given by $(\mathcal{H}, Q) \mapsto (\mathcal{H}, Q^2)$. The Hamiltonian is the square of the supercharge Q.

Example 1.14. Topological supersymmetric theories are $(\mathcal{H}, Q) = (\mathcal{H}, 0)$.

Example 1.15. Take M a compact oriented Riemannian manifold and $\mathcal{H} = \Omega^*(M)$ the $\mathbb{Z}/2$-graded vector space of square-integrable differential forms. Then $Q = d + d^*$ is the Hodge-de Rham operator. This is a compact supersymmetric quantum mechanical system. We have $Q^2 = \Delta$ the Laplacian, recovering the de Rham example above. There is a second square root of the Laplacian, $Q' = i(d - d^*)$, which is also a compact supersymmetric quantum mechanical system. ($\mathcal{N} = 2$ supersymmetry, Dan refers to Witten's paper on Morse theory [**?**].)

Example 1.16. Let M be a Riemannian spin manifold and $\mathcal{H} = \Gamma(S)$ the space of L^2-sections of the spinor bundle. Then $Q = \not{D}$ is the Dirac operator.

Example 1.17. M spin and $E \to M$ a vector bundle with hermitian metric and metric connection. Then $H = \Gamma(S \otimes E)$ and $Q = \not{D}_E$ is the twisted Dirac operator.

See 1.3 for a picture of a supersymmetric Q. We get superconfigurations. The topology is such that eigenspaces can still merge and split, but now they can also change parity.

Theorem 1.18 (Hohnhold-Stolz-Teichner)**.** $\mathrm{sConf}(\mathcal{H}) \simeq BU \times \mathbb{Z}$ *So* $\mathsf{SQM} \simeq \mathrm{sConf}(\mathcal{H})/PU^{\pm}(\mathcal{H})$. *This classifies twisted equivariant K-theory. Furthermore* $\mathrm{sConf}(\mathcal{H}) \to \mathbb{Z}$, $Q \mapsto \mathrm{sdim}(\ker(Q))$ *is the super-dimension of the ground states.*

Remark 1.19. This is also the Witten index of the supersymmetric quantum mechanical system, i.e. $\mathrm{sdim}(\ker(Q)) = \mathrm{Tr}\left((-1)^F e^{-tQ^2}\right)$ where F is the fermion number operator.

1.4 RG flow categories

Given a flow $\mathbb{R}_{>0} \times X \to X$, define the flow category Flow_X with objects fixed points of the action and morphisms are broken flow lines from x to y. This is a topological category. Picture!

Theorem 1.20 (Cohen-Jones-Segal)**.** *When X is a compact Riemannian manifold and the flow is the gradient flow of a Morse function, then*

$$B\mathrm{Flow}_X \simeq X$$

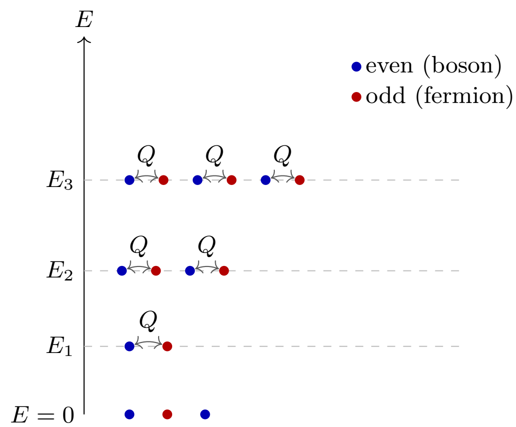


Figure 1: The energy levels of a supersymmetric Hamiltonian $H = Q^2$ have a symmetry for $E > 0$, i.e. if v is an eigenvector of H with nonzero eigenvalue, then Qv is an eigenvector with the same eigenvalue; Q pairs states of opposite parity (blue = even/boson, red = odd/fermion). When deforming a supersymmetric Hamiltonian, $E = 0$ eigenspaces can only be lifted pairwise to positive energy levels. The super-dimension of the $E = 0$ eigenspace stays constant.

Idea: Apply this to study QFT. This is very explicit for QM and SQM.

Example 1.21 (Flow category for QM). $\mathrm{Flow}_{\mathrm{Conf}(\mathcal{H})}$ has objects the topological theories $(V, 0)$ with V a finite dimensional inner product space. A morphism is a pair of an isometric embedding $V \hookrightarrow W$ and H an invertible self-adjoint operator on $V^{\perp}$.

This category has a weakly initial object $\{0\}$, where the unique morphism to V is given by H. The category is contractible.

Example 1.22 (Flow category for SQM). Objects are $\mathbb{Z}/2$-graded finite dimensional inner product spaces. Morphisms are odd operators Q on the complement.

Say something about $S^{-1}S$.

Theorem 1.23 (Quillen)**.** $B\mathrm{Flow}_{\mathrm{sConf}(\mathcal{H})} \simeq \mathbb{Z} \times BU$

2 Lecture 2

Recall $\mathsf{QM} \simeq \mathrm{Conf}(\mathcal{H})/PU(\mathcal{H})$ and $\mathsf{SQM} \simeq \mathrm{sConf}(\mathcal{H})/PU^{\pm}(\mathcal{H})$.

$\mathrm{Flow}_{\mathrm{Conf}(\mathcal{H})}$ has a weakly initial object $\{0\}$, so its classifying space is contractible. For $\mathrm{Flow}_{\mathrm{sConf}(\mathcal{H})}$, $\mathrm{Hom}(V, W) \neq 0$ if and only if they represent the same element in $\pi_0 KU$.

Remark 2.1. The Bott class is...

2.1 A definition of QFT

There are probably many definitions of QFT, but here we use a TQFT/Segal CFT-like definition.

Definition 2.2. A d-dimensional QFT with geometry $\mathcal{F}$ is a symmetric monoidal functor

$$Z : \mathrm{Bord}_d(\mathcal{F}) \to \mathrm{Vect}$$

where $\mathrm{Bord}_d(\mathcal{F})$ is the bordism category of oriented $d-1$-dimensional manifolds with geometry $\mathcal{F}$.

$\mathcal{F}$ is a sheaf or stack on the category of d-dimensional manifolds, e.g. Riemannian metrics, spin structures, etc. Vect is your favorite convenient category of (functional analytic) vector spaces. Some aspects of the definition one might ask for:

- want Z to depend continuously/smoothly on various input parameters, e.g. on the metric if $\mathcal{F}$ is the sheaf of Riemannian metrics.
- objects & morphisms in $\mathrm{Bord}_d(\mathcal{F})$ can have internal automorphisms, e.g. bundle automorphisms if $\mathcal{F}$ is the stack of G-bundles.
- bordisms should be collared for well-defined composition
- $\mathrm{Bord}_d(\mathcal{F})$ could be unital or not (cylinder of length zero)
- allow supermanifolds as bordisms and super vector spaces as target; then bordisms need to be equipped with a spin structure
- vector spaces should have a topology, maps should be continuous
- $\otimes$ needs to be topologized, e.g. one can take Fréchet spaces and completed projective tensor product

Useful structures on this flavor of QFT can be built from the geometry of $\mathcal{F}$.

Example 2.3. There is a forgetful functor $\mathrm{Bord}_d(\mathcal{F}) \to \mathrm{Bord}_d$. The induced map on functor categories is

$$\mathsf{TQFT}_d \to \mathsf{QFT}_d(\mathcal{F})\,.$$

The image are the topological field theories (that do not depend on the geometry).

Example 2.4. $\mathcal{F} = \mathrm{Riem}$ Riemannian metrics. Then $\mathbb{R}_{>0}$ acts on $\mathrm{Bord}_d(\mathcal{F})$ inducing the RG flow on $\mathsf{QFT}_d(\mathrm{Riem})$.

2.2 Reflection Positivity

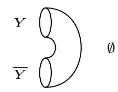
The euclidean analogue of unitarity of a QFT is reflection positivity. This can be axiomatized:

$$Z : \underset{C_2\curvearrowright}{\mathrm{Bord}_d} \to \underset{C_2\curvearrowright}{\mathrm{Vect}}$$

1. There is a reflection structure on $\mathrm{Bord}_d(\mathcal{F})$ generated by orientation reversal.
2. On Vect there is the complex conjugation C_2-action.
3. The field theories are required to preserve these involutions, i.e. we have to specify equivariance data $\overline{Z(Y)} \cong Z(\overline{Y})$. More formally, there are coherence conditions which amount to Z beinga homotopy fixed point for the diagonal C_2-action on functors.
4. positivity property: For $Y \in \mathrm{Bord}_d$ there is a map

$$\overline{Z(Y)} \otimes Z(Y) \cong Z(\overline{Y}) \otimes Z(Y) \to \mathbb{C}\,,$$

obtained by applying Z to the cobordism



We require this to be a positive definite.

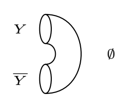
We now also want to add fermions to the theory and for that we have to talk about spin. Along the same lines as reflection positivity, we want to implement or impose a spin-statistics theorem.

$$Z : \underset{C_2 \times BC_2 \circlearrowright}{\mathrm{Bord}_d^{\mathrm{Spin}}} \to \underset{C_2 \times BC_2 \circlearrowright}{\mathrm{sVect}}$$

1. There is a reflection structure on $\mathrm{Bord}_d^{Spin}(\mathcal{F})$ generated by orientation reversal and there is a BC_2 action generated by the spin flip.
2. On sVect there is the complex conjugation C_2-action and there is a BC_2 action $(-1)^F$.
3. The field theories are required to preserve these involutions, i.e. we have to specify equivariance data $\overline{Z(Y)} \cong Z(\overline{Y})$. More formally, there are coherence conditions which amount to Z beinga homotopy fixed point for the diagonal C_2-action on functors. This is a spin reflection sructure.
4. positivity property: For $Y \in \mathrm{Bord}_d$ there is a map

   $$\overline{Z(Y)} \otimes Z(Y) \cong Z(\overline{Y}) \otimes Z(Y) \to \mathbb{C}\,,$$

   obtained by applying Z to the cobordism

   We require this to be a positive definite.
5. If $\mathcal{F}$ is present; need geometric involution that replaces the orientation reversal and we might only require positivity for some objects and need natural evaluation.

Remark 2.5. An action of a group G on a category $\mathcal{C}$ is a functor $BG \to \mathrm{Cat}$ that maps $*$ to $\mathcal{C}$. Additionally this supplies a map $\phi : G \to \mathrm{Fun}(\mathcal{C}, \mathcal{C})$ and natural isomorphisms $\phi_g \circ \phi_h \cong \phi_{gh}$.

If G is abelian, BG is an abelian 2-group and we can make sense of an action of BG on $\mathcal{C}$ as a functor (of 2-categories) $B^2G \to \mathrm{Cat}$ mapping $* \to \mathcal{C}$. This is the data of a homomorphism from G to natural transformations of the identity functor. $g \mapsto (\theta_g : \mathrm{id}_{\mathcal{C}} \Rightarrow \mathrm{id}_{\mathcal{C}})$, i.e. for every object an endomorphism $\theta_g : c \to c$.

2.3 Unpacking the definition in dimension 1

First, let's unravel what assignments a topological field theory in dimension one makes:

$$\begin{aligned} \mathrm{Bord}_1 &\longrightarrow \mathrm{Vect} \\ \mathrm{pt}_+ &\longmapsto V_+ \\ \mathrm{pt}_- &\longmapsto V_- \cong V^\vee \cong \overline{V} \\ \underset{\pm \;\; \mp}{\frown} &\longmapsto V_\pm \otimes V_\mp \to \mathbb{C} \\ \overset{\pm \;\; \mp}{\smile} &\longmapsto \mathbb{C} \to V_\pm \otimes V_\mp \\ \bigcirc &\mapsto \mathrm{Tr}(\mathrm{id}_{V_\pm}) \end{aligned}$$

Remark 2.6. The functoriality imposes that the circle is mapped to the trace.

Want that $\overline{V} \otimes V \cong V_- \otimes V_+ \to \mathbb{C}$ is an inner product *and V should be a Hilbert space.* If we add spin: two spin structures on the circle: a bounding one and a nonbounding one:

$$\begin{aligned} \mathrm{Bord}_1^{\mathrm{Spin}} &\longrightarrow \mathrm{sVect} \\ S_b^1 &\longmapsto \mathrm{Tr}(\mathrm{id}_V) \\ S_{nb}^1 &\longmapsto \mathrm{str}(\mathrm{id}_V) \end{aligned}$$

Remark 2.7. To explain why the nonbounding circle is assigned the super trace, we have to unravel spin structures in dimension 1 and functoriality.

We now want to add Riemannian metrics. There is now a family of interval bordisms $I_t = \overset{+}{\underset{+}{|}}\, {}^t$ parametrized by their length t. Since $I_{t_1+t_2} = I_{t_1} \circ I_{t_2}$, Z maps this to a 1-parameter semigroup $Z(I_t) : V \to V$ which has an infinitesimal generator H. Note that the bordism category is no longer fully dualizable. We will only get $V_- \subset V^\vee$ via the evaluation of length t.

$$\begin{aligned} \mathrm{Bord}_d^{\mathrm{Riem,Spin}} &\longrightarrow \mathrm{sVect} \\ \overset{-}{\underset{+}{|}}\, {}^t &\longmapsto e^{-tH} \\ \underset{\pm \;\; \mp}{\frown}\, {}^t &\longmapsto \langle -, e^{-tH} - \rangle \\ \overset{\pm \;\; \mp}{\smile}\, {}^t &\longmapsto (\mathbb{C} \ni 1 \mapsto e^{-tH} \in V_\pm \otimes V_\mp) \\ S_b^1 &\longmapsto Tr(e^{-tH}) \\ S_{nb}^1 &\longmapsto \mathrm{str}(e^{-tH}) \end{aligned}$$

Going to super $N = 1$; new θ parameter. There is a 1-parameter super semigroup $\mathbb{R}^{1|1}_{\geq 0}$. This has an infinitesimal odd generator Q with $Q^2 = H$.

$$\overset{-}{\underset{+}{|}}\, {}^{(t,\,\theta)} \longmapsto e^{-tH+\theta Q}$$

Remark 2.8. There is a super Lie group $\mathbb{R}^{1|1}$ with multiplication given by

$$(t_1, \theta_1)(t_2, \theta_2) = (t_1 + t_2 + \theta_1\theta_2, \theta_1 + \theta_2)\,.$$

Its super Lie algebra has an odd generator $Q = \partial_\theta$ and an even generator $H = \partial_t$ with $[Q, Q] = 2H$.

Remark 2.9. The super trace $\mathrm{str}(e^{-tH+\theta Q})$ is constant and equal to index of Q.

Example 2.10. Quantum mechanics on a compact oriented Riemannian manifold M. $V = C^\infty(M) = \overline{V}$, $\langle f, g\rangle = \int \overline{f} g \, d\mathrm{vol}_g$ $H = \Delta$.

Example 2.11. Supersymmetric quantum mechanics on a compact Riemannian spin manifold. $V = \Gamma(M; S)$ and $Q = D$ the Dirac operator.

2.4 Aspects of 2d theories

2d Riemannian theories

$$\begin{array}{ccc}
\text{Annuli} \hookrightarrow \mathrm{Bord}_2(\mathrm{Riem}) & \longrightarrow & \mathrm{Vect} \\
S^1_l & \longmapsto & V_l \\
\mathrm{Cyl}_{t,x} & \longmapsto & e^{-tH+ixP} : V_l \to V_l
\end{array}$$

$x \mod 2\pi/l$ is the rotation of the boundary circle on the cylinder of length t. The cylinders or annuli define another semigroup with $(x_1, t_1) + (x_2, t_2) = (x_1 + x_2, t_1 + t_2)$.

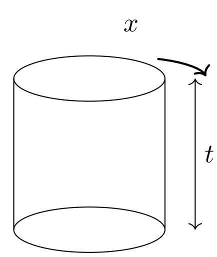


Figure 2: The cylinder $\mathrm{Cyl}_{t,x}$: a bordism from S^1_l to itself of length t, with the top boundary circle glued back with a rotation by $x \in 2\pi\mathbb{R}/l\mathbb{Z}$.

Introduce $\frac{x+it}{l} = \tau \in \mathbb{H}/\mathbb{Z}$ $x + it \in \mathbb{H}/l\mathbb{Z}$. We get a semigroup representation of $\mathbb{H}/\mathbb{Z}$, where $U(1)$ acts unitarily. There is an infinitesimal generator P such that $x \mapsto e^{ixP}$ defines the $U(1)$-action on V_l. The spectrum of P must be contained in $\frac{2\pi}{l}\mathbb{Z}$ since $U(1)$-actions can only have integral eigenvalues. Similarly for t, there is an infinitesimal generator H. Define

$$H = \frac{2\pi}{l}(L_0 + \overline{L}_0)\,,$$
$$P = \frac{2\pi}{l}(L_0 - \overline{L}_0)\,.$$

Then with $q = e^{2\pi i\tau}$ we can write

$$e^{-tH+ixP} = q^{L_0}\overline{q}^{-\overline{L}_0}$$

Adding spin just adds a grading involution.?

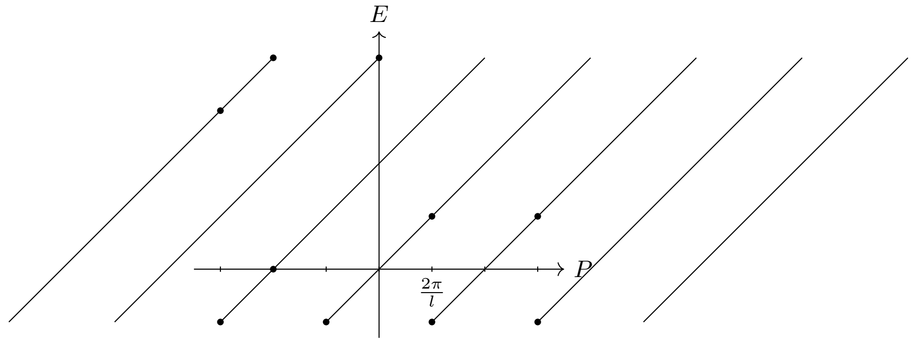


Figure 3: Energy states for a $2d$ Hamiltonian. Diagonals are eigenvalues of $\overline{L}_0$. Each diagonal line is a fixed value of $\overline{L}_0$, with L_0 increasing along the line.

2d $\mathcal{N} = 0|1$-theories Adding $0|1$ supersymmetry adds (x, t, θ) as parameters of annuli. Restricting to super annuli we obtain a representation of the super semigroup $\mathbb{H}^{2|1}/l\mathbb{Z}$ in trace-class operators. There are infinitesimal generators such that

$$\mathrm{Cyl}_{t,x,\theta} \mapsto e^{-tH+ixP+\theta Q} : V_l \to V_l \,.$$

$$e^{-tH+ixP+\theta Q} = q^{L_0}\overline{q}^{\overline{L_0}}e^{\theta Q}$$

Here, H, P are as before and $Q^2 = \overline{L}_0$. Note that there is no odd generator which squares to L_0.

Remark 2.12. In two dimensions with $\mathcal{N} = (0|1)$ supersymmetry, the multiplication of the super Lie group $\mathbb{R}^{2|0,1}$ is given by

$$(\tau_1, \overline{\tau}_1, \theta_1)(\tau_2, \overline{\tau}_2, \theta_2) = (\tau_1 + \tau_2, \overline{\tau}_1 + \overline{\tau}_2 + \theta_1\theta_2, \theta_1 + \theta_2)\,.$$

Its super Lie algebra has an odd generator $Q = \partial_\theta$ and two even generators $L_0 = \partial_\tau . \overline{L_0} = \partial_{\overline{\tau}}$ with $[Q, Q] = 2\overline{L_0}$ and all other commutators vanishing.

Only for the trivial spin structure (odd, periodic, Ramond)?x

Configuration space picture of (H, P) See figure 4. Each energy level of H is a $U(1)$-representation, i.e. has a fixed weight. Also negative eigenvalues, but bounded below. $H - P = Q^2 \geq 0$ and hence eigenvalues lie above the diagonal $H = P$. Diagonal and antidiagonal lines are eigenvalues of L_0 and $\bar{L}_0$. Supersymmetry: dimension of even and odd spaces on antidiagonals are the same.

2.5 Lecture 3

Recall from last time:

- $$\mathbb{R}^{1|1} \hookrightarrow \mathrm{Bord}_{1|1} \to \mathrm{sVect}$$

  determines $e^{-tH+\theta Q}$ acting on $\mathcal{H}$, $Q^2 = H$. H, Q are self-adjoint by reflection positivity.

- For each $l > 0$ have annuli of circumference l and

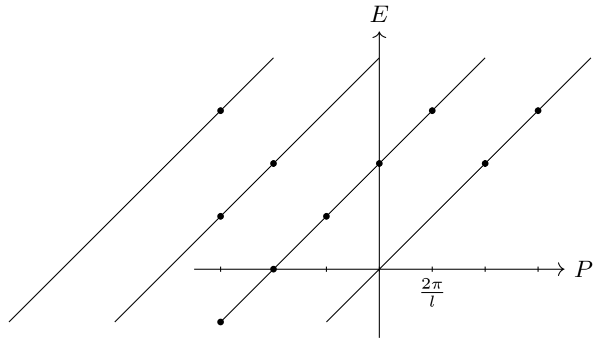


Figure 4: Energy states for a 2d $N = 0|1$ Hamiltonian. Diagonals are eigenvalues of $\overline{L}_0$. Each diagonal line is a fixed value of $\overline{L}_0$, with L_0 increasing along the line. All eigenvalues lie above the diagonal. For all diagonals above $H = P$, Q is a bijection between even and odd eigenstates.

- $$\mathbb{H}^{2|1}/l\mathbb{Z} \hookrightarrow \mathrm{Bord}_{2|1} \to \mathrm{sVect}$$

  determines a trace-class semigroup $e^{-tH+ixP+\theta Q} = q^{L_0}\overline{q}^{\overline{L_0}} e^{\theta\sqrt{(\frac{2\pi}{l})}\overline{G_0}}$ where $\overline{G_0} = \sqrt{\frac{2\pi}{l}}Q$. $Q^2 = \frac{1}{2}(H-P)$. Again, H, P, Q are self-adjoint.

  $L_0 = \frac{l}{4\pi}(H+P)$ and $\overline{L_0} = \frac{l}{4\pi}(H-P)$.

Dan says the notation is borrowed from CFT. We also recall figure 3. P generates a circle action and has eigenvalues in $\frac{2\pi}{l}\mathbb{Z}$. H and P are simultaneously diagonalizable since they commute. Supersymmetry forbids the region below $H = P$-diagonal. On the diagonals $H - P = 2\lambda$, Q^2 acts by λ.

2.5.1 Partition functions

The Ramond-Ramond (odd-odd, nb-nb) partition function is the value of a QFT $\mathrm{Bord}_{2|1} \to \mathrm{sVect}$ on the super torus with underlying spin structure the RR/odd spin structure.

Fact: The value on the torus is the super-trace of the value on the cylinder. So the partition function is $Z(l, \tau, \overline{\tau}) = \mathrm{str}(e^{-th+ixP+\theta Q})$ as a function on $\mathbb{R}_{>0} \times \mathbb{H}^{2|1}$.

Remark 2.13. There are four spin structures on the torus. Three of them are permuted under the mapping class group action. The RR-spin structure is the fixed point. For the other ones the partition function is the trace and not the supertrace.

For formal reason, the partition function does not depend on θ. The function is also independent of $\overline{\tau}$:

$$\begin{aligned}
\partial_{\overline{\tau}} \,\mathrm{str}(q^{L_0}\overline{q}^{\overline{L_0}}) &= -2\pi i\,\mathrm{str}(\overline{L_0} q^{L_0}\overline{q}^{\overline{L_0}}) \\
&= -2\pi i\,\mathrm{str}(\overline{G_0}^2 q^{L_0}\overline{q}^{\overline{L_0}}) \\
&= -\pi i\,\mathrm{str}([\overline{G_0}, \bar{G}_0 q^{L_0}\overline{q}^{\overline{L_0}}]) \\
&= 0
\end{aligned}$$

The argument for integrality:

$$\begin{aligned}
\operatorname{str}(q^{L_0}\overline{q}^{\overline{L_0}}) &= \operatorname{str}(q^{L_0}|_{\ker(\bar{L}_0)}) \\
&= \sum_k \operatorname{str}(q^{L_0}|_{\ker(\bar{L}_0)\cap \operatorname{Eig}_k(p)}) \\
&= \sum_{k>-N} q^k \operatorname{sdim}(\ker(\bar{L}_0)\cap \operatorname{Eig}_k(p))
\end{aligned}$$

For the first step, $G_0^2 = \overline{L_0}$ and the nonzero eigenspaces of $\bar{L}_0$ do not contribute. This is an integral power series in q and hence independent of $\bar{\tau}$ and also of l (!) since the super dimension does not jump under continuous deformation of l.

Remark 2.14. The index or superdimension is continuous. The ordinary trace is not continuous and the trace can hence depend on l.

The argument for modularity: Consider the torus $T_{l,\tau} := \mathbb{C}/(l\mathbb{Z}\oplus l\tau\mathbb{Z})$ where $l\in\mathbb{R}_{>0}, \tau\in\mathbb{H}$. For $\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in \mathrm{SL}_2(\mathbb{Z})$ we get

$$T_{|c\tau+d|l,\frac{a\tau+b}{c\tau+d}} \cong T_{l,\tau}\,.$$

Claim: this also lifts to an isomorphism between super tori; this involves the metaplectic group, the double cover of $\mathrm{SL}_2(\mathbb{Z})$. Putting this together we get:

Theorem 2.15. *The partition function of a 2d $N = 0|1$ EFT is a weight zero modular form, i.e.*

$$\begin{aligned}
Z(l,\tau,\bar{\tau}) &= Z(\tau)\in\mathcal{O}(\mathbb{H}) \\
Z(\frac{a\tau+b}{c\tau+d}) &= Z(\tau)
\end{aligned}$$

The weight (which is zero here) is also called central charge or anomaly.

2.5.2 Anomalies

Motivation from projective representation theroy. A G-representation is the same data as a functor (of stacks) $BG\to\mathrm{Vect}$. We get a projective G-representations from natural transformations

$$\begin{tikzcd} BG \arrow[r, bend left, "\alpha"{name=U}] \arrow[r, bend right, "\mathbb{1}"'{name=D}] & \mathrm{Bimod} \arrow[Rightarrow, from=D, to=U, "Z"'] \end{tikzcd}$$

For α $\otimes$-invertible this is a projective representation. Briefly, α determines a degree 2 cocylce $G\times G\to\mathbb{C}^\times$ and Z is a representation of the extension G^α of G by $\mathbb{C}^\times$ classified by α.

Definition 2.16. An anomalous QFT with anomaly α is a *(op)lax* monoidal natural transformation

$$\begin{tikzcd} \mathrm{Bord}_d(\mathcal{F}) \arrow[r, bend left, "\alpha"{name=U}] \arrow[r, bend right, "\mathbb{1}"'{name=D}] & \mathrm{Bimod} \arrow[Rightarrow, from=D, to=U, "Z"'] \end{tikzcd}$$

and α is $\otimes$-invertible.

Unpacking this: for a $d-1$ manifold Y we get $\alpha(Y)$ and algebra and ${}_{\mathbb{C}}Z(Y)_{\alpha(Y)}$ bimodule. A bordism $\Sigma : Y_{in} \to Y_{out}$ is assigned an invertible bimodule ${}_{\alpha(Y_{out})}\alpha(\Sigma)_{\alpha(Y_{in})}$ and a bimodule map

$$Z(\Sigma) : \alpha(\Sigma) \otimes_{\alpha(Y_{in})} Z(Y_{in}) \to Z(Y_{out})\,.$$

This comes from the following diagram:

$$\begin{tikzcd}
\mathbb{1} \arrow[r] \arrow[d, "Z_{Y_{in}}"'] & \mathbb{1} \arrow[d, "Z_{Y_{out}}"] \\
\alpha(Y_{in}) \arrow[r, "\alpha(\Sigma)"'] \arrow[ur, phantom, "\Nearrow Z(\Sigma)"] & \alpha(Y_{out})
\end{tikzcd}$$

Remark 2.17. The reason to require α to be (op)lax is as follows: Let's say we are working with TQFTs where $\mathcal{F} = *$, in which case Bord_d is fully dualizable. If $F, G : \mathcal{C} \to \mathcal{D}$ are symmetric monoidal functors and $\mathcal{C}$ is fully dualizable, then any (strong) monoidal natural transformation $Z : F \Rightarrow G$ must be an equivalence. (This is a nice exercise.) This would force $\alpha \cong \mathbb{1}$.

Reflection Positivity Again going to be data+property. In examples, the datum is a star-structure on the algebras $\alpha(Y)$. The positivity condition is the C^*-condition.

2.6 Exercise Session

Exercise 2.18. For Q odd self-adjoint. Show that $\mathrm{str}(e^{-tQ^2}) = \mathrm{Ind}(Q) = \mathrm{sdim}(\ker(Q))$.

Exercise 2.19. Show

$$e^{-tQ^2+\theta Q} e^{-t'Q^2+\theta' Q} = e^{-(t+t'+\theta\theta' \pm \theta\theta')Q^2 + (\theta+\theta')Q}\,.$$

3 Lecture 4

Recall from last time: An anomalous QFT with anomaly is a natural transformation $Z : \alpha \to \mathbb{1}$.

Remark 3.1. Often, anomalies are restrictions of $d+1$-dimensional theories.

Example 3.2.

$$\begin{tikzcd}
\mathrm{Bord}_{1|1} \arrow[r, bend left, "\mathrm{Cl}_n"{name=U}] \arrow[r, bend right, "\mathbb{1}"'{name=D}] & \mathrm{sBimod}
\arrow[from=D, to=U, Rightarrow, "Z"']
\end{tikzcd}$$

The twist $\alpha = \mathrm{Cl}_n$:

$$\begin{aligned}
\mathrm{pt}_+ &\longmapsto \mathrm{Cl}_n \\
\mathrm{pt}_- &\longmapsto \mathrm{Cl}_n^{\mathrm{op}} \\
\underset{\pm\ \ \mp}{\frown} &\longmapsto {}_{\mathrm{Cl}_n \otimes \mathrm{Cl}_n^{\mathrm{op}}}(\mathrm{Cl}_n)_{\mathbb{C}} \\
\overset{\mp}{\underset{\pm}{|}} &\longmapsto \mathrm{id}_{\mathrm{Cl}_n}
\end{aligned}$$

Reflection positivitiy imposes the data of $* : \overline{\mathrm{Cl}_n} \cong \mathrm{Cl}_n^{\mathrm{op}}$. This we take to be the standard C^*-structure.[1]

Any Cl_n-anomalous field theory Z assigns

$$\begin{aligned}
\mathrm{pt}_+ &\longmapsto \mathcal{H}_{\mathrm{Cl}_n} \\
\mathrm{pt}_- &\longmapsto \overline{\mathcal{H}}_{\overline{\mathrm{Cl}_n}} \\
\underset{\pm\ \ \mp}{\frown} &\longmapsto (\overline{\mathcal{H}} \otimes_{\mathrm{Cl}_n^{\mathrm{op}}} \mathcal{H} \to \mathbb{C}) \\
\underset{+}{\overset{+}{\Big|}}\ {\scriptstyle (t,\theta)} &\longmapsto (e^{-tH+\theta Q} : \mathcal{H} \to \mathcal{H}) \\
\bigcirc &\longmapsto \mathrm{str}_{\mathrm{Cl}_n}(e^{-tH+\theta Q})
\end{aligned}$$

There is a C_2 action on the bordism category which heuristically is $(t, \theta) \mapsto (-t, i\theta)$.[2]

Theorem 3.3 (Stolz-Teichner)**.** *The collection of Cl_n-anomalous theories gives geometric cocylces for KU^n.*

Using the action, one can also get Real K-theory KR^n. Should be explained better.

Sketch. Q is an odd Cl_n-linear unbounded self-adjoint operator with some properties. Q classifies all the data above. The collection of these represents the nth space of the K-theory spectrum. □

We still have to understand the connecting maps to get an Ω-spectrum.

Example 3.4 (Thom class). Let V an inner product space. Take $\mathrm{Cl}(V)$ the anomalous theory determined by $\mathcal{H} = \mathrm{Cl}(V)$ and $Q_v = cl_v = v \cdot - : \mathcal{H} \to \mathcal{H}$ for $v \in V$. This is a family (of field theories) indexed by $v \in V$ and

$$[\sigma_V] \in KR^V(V, V \setminus 0)$$

and represents the suspension class. For $v \neq 0$ this operator is invertible and represents the zero class in K-theory; another way to think about this is that the family of field theories is compactly supported since $e^{-tQ_v^2+\theta Q_v}$ goes to zero for $v \to \infty$.

Examples in 2d

Example 3.5. We define a chiral anomaly theory

$$\mathrm{Bord}_{2|1} \xrightarrow{\mathrm{Fer}_V} \mathrm{sBimod}$$

$$\begin{aligned}
\bigcirc &\longmapsto \mathrm{Cl}(\Gamma(S^1, \mathbb{S} \otimes V))) \\
\Sigma &\longmapsto \ker(\not{D}_V) \underset{\text{Lagrangian}}{\subset} \Gamma(\partial\Sigma, \mathbb{S} \otimes V) \\
&\quad \text{Fock space gives } \mathrm{Fer}_V(\Sigma) \\
\partial\Sigma = \emptyset &\longmapsto \mathrm{Pf}(\not{D}_V)
\end{aligned}$$

This is chiral, i.e. $Q =$, can be made $\mathcal{N} = 0|1$ supersymmetric.

[1] Here, the signs are delicate because there are conflicting sign convenctions in the C*-literature versus Koszul sign rule.

[2] THeo: In condensed matter this weird C_2/C_4-action is called fractionalization.

We have $\mathrm{Fer}(V) \otimes \mathrm{Fer}(W) \cong \mathrm{Fer}(V \oplus W)$.

Theorem 3.6 (Stolz-Teichner)**.** *Using the flat Riemannian version of* $\mathrm{Bord}_{2|1}$, $\mathrm{Fer}_n = \mathrm{Fer}(\mathbb{R}^n)$ *is 48-periodic.*

Conjecture 3.7. *There exists a fully extended version of* Fer_V *that is* $24^2 = 576$ *periodic.*

Conjecture 3.8. *In the fully extended case, taking canonical Fock modules for* Fer_V *there is a* V*-family of supersymmetric deformations and this represents the suspension class in* $TMF^V(V, V \setminus 0)$.

3.1 Equivariance

Assigning trivial G-bundles with trivial connection yields:

$$\mathrm{Bord}_d \longrightarrow \mathrm{Bord}_d(B_\nabla G)$$

On field theories there is an induced restriction functor from field theories to field theories with G-symmetry.

Example 3.9. 1d. $G \to O(V)$ representation. This equipps $\mathrm{Cl}(V)$ with G-symmetry. There is a central extension $\mathrm{Pin}^c(V) \to O(V)$ which can be pulled back to $\hat{G} \to G$

$$\begin{tikzcd}
\mathrm{Bord}_{1|1}(B_\nabla G) \arrow[r, bend left, "\mathrm{Cl}(V)"{name=U}] \arrow[r, bend right, "\mathbb{1}"'{name=D}] & \mathrm{sBimod}
\arrow[from=D, to=U, Rightarrow, "Z"']
\end{tikzcd}$$

$$\begin{aligned}
\mathrm{pt}_+ &\longmapsto \mathcal{H}_{\mathrm{Cl}(V)} \circlearrowleft G \\
\overset{+}{\underset{+}{\big|}}\ {\scriptstyle (t,\,\theta)} &\longmapsto e^{-tH+\theta Q}\rho(g) \\
P_g \to \bigcirc\ {\scriptstyle \mathrm{hol}_{S^1} = g} &\longmapsto \mathrm{str}_{\mathrm{Cl}(V)}(e^{-tH+\theta Q}\rho(g)) \in \Gamma(G/G; \mathcal{L})
\end{aligned}$$

Q needs to commute with the G-action. The connection gives a parallel transport.

Example 3.10. $\mathcal{H} = \mathrm{Cl}(V)_{\mathrm{Cl}(V)} \circlearrowleft \hat{G}$, Q_v Clifford multiplication by v as in suspension class. We get a. G-equivariant V-family of odd operators. We obtain

$$\mathrm{Th}(\rho) \in KR_G^{\tau_\rho}(V, V \setminus 0)$$

where the twist of KR is classified by the collection $(w_1(\rho), W_3(\rho), \dim(\rho))$.

The classes $\mathrm{Th}(\rho)$ for varying $\rho : G \to O(V)$ contain all the orientations of K-theory you might want.

- $M\,\mathrm{Spin} \to KO$
- $MU \to KU$

3.2 Examples in 2d

The degree 4 cohomology generator in $O(V)$ represents a smooth 2-group extension[3]:

$$\begin{tikzcd} \hat{G} \arrow[r] \arrow[d] & \mathrm{Tring} \arrow[d] \\ G \arrow[r] & O(V) \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Bord}_{2|1}(B_\nabla G) \arrow[r, bend left, "\mathrm{Fer}(\rho)"] \arrow[r, bend right, "\mathbb{1}"'] & \mathrm{sBimod} \end{tikzcd}$$

$$\begin{aligned} P_g \to S^1 &\longmapsto \mathrm{Cl}(\Gamma(S^1, \mathbb{S} \otimes P_g \times_{O(V)} V)) \\ P \to \Sigma &\longmapsto \mathrm{Fock}(\mathrm{Lag}-\mathrm{corr}) \\ P_{g,h} \to T^2 &\longmapsto \text{Pfaffian line over moduli of } G\text{-bundles, } \theta\text{-line} \end{aligned}$$

The holonomy classifies G-bundles $P_g \to S^1$ and over T^2 there are two group elements have to commute. The mapping class group action of $\Gamma < \mathrm{SL}_2(\mathbb{Z})$ stabilizing (g,h) on $P_{g,h} \to T^2$ gives level structures?

The Tring group acts on the spinor bundle...

For anomalous theories Z of this type, get that $\mathcal{H}_g = Z(P_g \to S^1) \circlearrowleft \mathrm{Fer}_V(P_g \to S^1)$ is a projective representation of $\mathrm{Aut}\widehat{(P_g \to S^1)}$, the central extension of a (twisted) loop group. We get a *canonical* anomalous theory from vacuum/spinor rep of $L\widehat{\mathrm{Spin}}(n)$.

Conjecture 3.11. *A fully extended version of* Fer_ρ *comes with a Thom class in twisted equivariant* TMF

$$\mathrm{Th}(\rho) \in TMF_G^{\tau_\rho}(V, V \setminus 0)\,.$$

This should in particular determine the string orientation.

Remark 3.12. Lurie proposes 2-equivariant TMF, the same 2 as in 2-group above. This has a universal property and can be used to produce a map from field theories to TMF.

The free fermion comes from a 3d theory, which assigns the eta invariant on 3-manifolds.

[3]The name "Tring" is a pun on the Pin-pun.