Kan Extension via Coends

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This note shows that the coend formula of Proposition 0.9 for Kan extensions remains valid for ∞-categories. In fact, we only perform elementary manipulations and deduce this result from [GHN17]. We start by recalling some notions.

Definition 0.1. Let $\mathcal{C}$ be an ∞-category. The (opposite of the) *twisted arrow category* $\mathrm{Tw}(\mathcal{C})^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ is the left (or: cocartesian) fibration classified by $\mathrm{Map}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Spc}$.

Unravelling the Grothendieck construction, the objects of $\mathrm{Tw}(\mathcal{C})^{\mathrm{op}}$ are given by tuples $(c_0, c_1, f \in \mathrm{Map}_{\mathcal{C}}(c_0, c_1))$, i.e. arrows $f : c_0 \to c_1$. A 1-morphism $f \to g$ is given by $\alpha : c_0' \to c_0$ and $\beta : c_1 \to c_1'$ and an equivalence $\beta_* \alpha^* f \simeq g$, i.e. 1-morphisms are commutative diagrams

$$\begin{tikzcd} c_0 \arrow[d, "f"] & c_0' \arrow[l, "\alpha"'] \arrow[d, "g"] \\ c_1 \arrow[r, "\beta"] & c_1' \, . \end{tikzcd} \tag{1}$$

Definition 0.2. The *end* of a functor $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ is the limit

$$\int_{c \in \mathcal{C}} F(c, c) := \lim \left(\mathrm{Tw}(\mathcal{C})^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{F} \mathcal{D} \right) . \tag{2}$$

The *coend* of a functor $F : \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ is the colimit

$$\int^{c \in \mathcal{C}} F(c, c) := \operatorname{colim} \left(\mathrm{Tw}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \xrightarrow{F} \mathcal{D} \right) . \tag{3}$$

The following is Proposition 5.1 in [GHN17].

Proposition 0.3. *Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. The space of natural transformations is given by the end*

$$\mathrm{Nat}(F, G) \simeq \int_{\mathcal{C}} \mathrm{Map}_{\mathcal{D}}(F(-), G(-)) . \tag{4}$$

This equivalence is natural in F and G.

Corollary 0.4. *Let $F : \mathcal{C} \to \mathrm{Spc}$ be a functor. Then there is an equivalence of functors $\mathcal{C} \to \mathrm{Spc}$:*

$$F(c) \simeq \int_{c' \in \mathcal{C}} F(c')^{\mathrm{Map}_{\mathcal{C}}(c, c')} .$$

Dually, we obtain

$$F(c) \simeq \int^{c' \in \mathcal{C}} \mathrm{Map}(c', c) \otimes F(c') ,$$

where $\otimes = \times$ is the self-tensoring of Spc *over* Spc.

Proof. We use the (contravariant) Yoneda embedding $\mathcal{C}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{C}, \mathrm{Spc})$. The Yoneda lemma implies

$$\begin{aligned} F(c) &\simeq \mathrm{Nat}(\mathrm{Map}_{\mathcal{C}}(c,-), F) \\ &\simeq \int_{c' \in \mathcal{C}} \mathrm{Map}_{\mathrm{Spc}}(\mathrm{Map}_{\mathcal{C}}(c,c'), F(c'))\,. \end{aligned}$$

To obtain the dual version, we compute using the Fubini theorem to interchange "integrals":

$$\begin{aligned} \mathrm{Nat}\left(\int^{c' \in \mathcal{C}} \mathrm{Map}_{\mathcal{C}}(c',-) \otimes F(c'), G\right) &\simeq \int_{c \in \mathcal{C}} \mathrm{Map}_{\mathrm{Spc}}\left(\int^{c'} \mathrm{Map}_{\mathcal{C}}(c',c) \otimes F(c'), G(c)\right) \\ &\simeq \int_{c'} \int_{c} \mathrm{Map}_{\mathrm{Spc}}\left(F(c'), G(c)^{\mathrm{Map}_{\mathcal{C}}(c',c)}\right) \\ &\simeq \int_{c'} \mathrm{Map}_{\mathrm{Spc}}\left(F(c'), \int_{c} G(c)^{\mathrm{Map}_{\mathcal{C}}(c',c)}\right) \\ &\simeq \int_{c'} \mathrm{Map}(F(c'), G(c')) \\ &\simeq \mathrm{Nat}(F,G)\,. \end{aligned}$$

Since this holds naturally for all G, the Yoneda lemma implies $F \simeq \int^{c' \in \mathcal{C}} \mathrm{Map}_{\mathcal{C}}(c',-) \otimes F$. ☐

In fact, the preceding corollary generalizes to any bicomplete target $\mathcal{D}$. Note that under this assumption there are tensoring and cotensoring functors

$$\otimes : \mathrm{Spc} \times \mathcal{D} \to \mathcal{D}\,, \qquad (-)^{-} : \mathcal{D} \times \mathrm{Spc}^{\mathrm{op}} \to \mathcal{D}$$

which satisfy the adjunctions $\mathcal{D}(d \otimes X, d') \simeq \mathrm{Spc}(X, \mathcal{D}(d,d')) \simeq \mathcal{D}(d, d'^{X})$.

Lemma 0.5. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that $\mathcal{D}$ is bicomplete. Then:*

$$\begin{aligned} F(c) &\simeq \int_{c' \in \mathcal{C}} F(c')^{\mathrm{Map}_{\mathcal{C}}(c,c')}\,, \\ F(c) &\simeq \int^{c' \in \mathcal{C}} \mathrm{Map}_{\mathcal{C}}(c',c) \otimes F(c')\,. \end{aligned}$$

Proof. We will verify only the first expression. We consider $\overline{F} : \mathcal{C} \times \mathcal{D}^{\mathrm{op}} \to \mathrm{Spc}$ obtained from postcomposing with the Yoneda embedding. Using the previous corollary and being careful about the occurring opposites:

$$\begin{aligned} \mathcal{D}^{\mathrm{op}}(F(c), d) = \overline{F}(c,d) &\simeq \int_{\mathcal{C} \times \mathcal{D}^{\mathrm{op}}} \mathcal{D}^{\mathrm{op}}(F(c'), d')^{\mathrm{Map}_{\mathcal{C}}(c,c') \times \mathcal{D}^{\mathrm{op}}(d,d')} \\ &\simeq \int_{\mathcal{D}^{\mathrm{op}}} \left(\int_{\mathcal{C}} \mathcal{D}^{\mathrm{op}}(F(c')^{\mathrm{Map}_{\mathcal{C}}(c,c')}, d')\right)^{\mathcal{D}^{\mathrm{op}}(d,d')} \\ &\simeq \int_{\mathcal{C}} \mathcal{D}^{\mathrm{op}}\left(F(c')^{\mathrm{Map}_{\mathcal{C}}(c,c')}, d\right) \\ &\simeq \mathcal{D}^{\mathrm{op}}\left(\int_{\mathcal{C}} F(c')^{\mathrm{Map}_{\mathcal{C}}(c,c')}, d\right) \end{aligned}$$

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Definition 0.6. Let $\gamma : \mathcal{C} \to \mathcal{D}$ be a functor. Let $\mathcal{E}$ be a bicomplete category. Then the restriction functor $\gamma^* : \mathrm{Fun}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})$ has a left adjoint $\gamma_!$ and a right adjoint γ_*. The *left (resp. right) Kan extension* of $F : \mathcal{C} \to \mathcal{E}$ is $\gamma_! F$ (resp. $\gamma_* F$).

Unravelling the universal property of this adjunction, the left Kan extension $\gamma_! F$ is the initial functor with a natural transformation $F \to \gamma^* \gamma_! F$.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\Searrow" near start] & \mathcal{E} \\
\mathcal{D} \arrow[ru, "\gamma_! F"']
\end{tikzcd}$$

Example 0.7. Presheaves are contravariant functors into $\mathcal{E} = \mathrm{Spc}$. The left (right) Kan extension along $\gamma^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is usually also denoted by $\gamma_!$ (γ_*). Let $c \in \mathcal{C}$ and consider the Yoneda presheaf $y_c = \mathrm{Map}_{\mathcal{C}}(-, c) : \mathcal{C}^{\mathrm{op}} \to \mathrm{Spc}$. There is a natural equivalence of functors:

$$\gamma_! y_c \simeq y_{\gamma(c)} \,. \tag{5}$$

To see this, we use the Yoneda Lemma for the chain of natural equivalences:

$$\begin{aligned}
\mathrm{Nat}(\gamma_! y_c, y_d) &\simeq \mathrm{Nat}(y_c, \gamma^* y_d) \\
&\simeq \gamma^* y_d(c) \\
&\simeq \mathrm{Map}_{\mathcal{D}}(\gamma(c), d) \\
&\simeq \mathrm{Nat}(y_{\gamma(c)}, y_d) \,.
\end{aligned}$$

Example 0.8. The dual of the previous example shows that for $\gamma : \mathcal{C} \to \mathcal{D}$ the left Kan extension of the functor $\mathrm{Map}_{\mathcal{C}}(c, -) : \mathcal{C} \to \mathrm{Spc}$ is given by

$$\gamma_! \mathrm{Map}_{\mathcal{C}}(c, -) \simeq \mathrm{Map}_{\mathcal{D}}(\gamma(c), -) \,. \tag{6}$$

Proposition 0.9 (Coend Formula for Kan extension)**.** *Let $F : \mathcal{C} \to \mathcal{E}$ and $\gamma : \mathcal{C} \to \mathcal{D}$ be functors. Let $\mathcal{E}$ be bicomplete. Then the left Kan extension can be computed as:*

$$\gamma_! F(d) \simeq \int^{\mathcal{C}} F(c) \otimes \mathrm{Map}_{\mathcal{D}}(\gamma(c), d) \,. \tag{7}$$

Dually, we have that the right Kan extension is

$$\gamma_* F(d) \simeq \int_{\mathcal{C}} F(c)^{\mathrm{Map}_{\mathcal{D}}(d, \gamma(c))} \,. \tag{8}$$

Note that we can reobtain Lemma 0.5 for $\gamma = \mathrm{id}$.

Proof. Write $F(-) = \int^{\mathcal{C}} \mathrm{Map}_{\mathcal{C}}(c, -) \otimes F(c)$. Using that the left adjoint $\gamma_!$ commutes with coends and using formula 6, we obtain

$$\gamma_! F(d) \simeq \int^{\mathcal{C}} \gamma_! \mathrm{Map}_{\mathcal{C}}(c, -)(d) \otimes F(c) \simeq \int^{\mathcal{C}} \mathrm{Map}_{\mathcal{D}}(\gamma(c), d) \otimes F(c) \,.$$

□

References

[GHN17] David Gepner, Rune Haugseng, and Thomas Nikolaus. Lax colimits and free fibrations in ∞-categories. *Doc. Math.*, 22:1225–1266, 2017.